Theory of Completeness for Logical Spaces

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Abstract. A logical space is a pair $(A, \mathcal{B})$ of a non-empty set A and a subset $\mathcal{B}$ of $\mathcal{P}A$. Since $\mathcal{P}A$ is identified with $\{0,1\}^A$ and $\{0,1\}$ is a typical lattice, a pair $(A, \mathcal{F})$ of a non-empty set A and a subset $\mathcal{F}$ of $\mathbb{B}^A$ for a certain lattice $\mathbb{B}$ is also called a $\mathbb{B}$-valued functional logical space. A deduction system on A is a pair (R, D) of a subset D of A and a relation R between A^* and A. In terms of these simplest concepts, a general framework for studying the logical completeness is constructed.

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Contents

1. Introduction

A **logical space** is a pair $(A, \mathcal{B})$ of a non-empty set A and a subset $\mathcal{B}$ of the power set $\mathcal{P}A$ of A. Let $\mathbb{T} = \{0, 1\}$. Then since $\mathcal{P}A$ is identified with $\mathbb{T}^A$ and $\mathbb{T}$ is a typical lattice, a pair $(A, \mathcal{F})$ of a non-empty set A and a subset $\mathcal{F}$ of $\mathbb{B}^A$ for a lattice $\mathbb{B}$ is also called a $\mathbb{B}$-valued **functional** logical space, if $\mathbb{B}$ has the smallest element 0 and the largest element 1 and is **non-trivial** in the sense that $\#\mathbb{B} \geq 2$.

Logical spaces naturally occur in logic. For instance, let A be the set of the formulas in PL (propositional logic). Then A is generated by the set X of variables, and each mapping ν of X into the set $\mathbb{T}$ of the truth values is as well-known extended to a mapping f_ν of A into $\mathbb{T}$, and we obtain a $\mathbb{T}$-valued functional logical space $\bigl(\mathsf{A}, \{\mathsf{f}_\nu\}\bigr)$. Also, let A be the set of the formulas in FPL (first-order predicate logic). Then A is determined by the sets C and X of the constants and variables, and each pair of mappings Φ and ν of C and X into each domain E of interpretation determines as well-known a mapping $\mathsf{f}_{\mathsf{E},\Phi,\nu}$ of A into $\mathbb{T}$, and we obtain a $\mathbb{T}$-valued functional logical space $\bigl(\mathsf{A}, \{\mathsf{f}_{\mathsf{E},\Phi,\nu}\}\bigr)$. Thus, a logical space is a model of a pair of the set of the sentences (as for FPL, not necessarily closed formulas) of a logic and the family of functions which assign the sentences their truth values in $\mathbb{T}$. Replacing $\mathbb{T}$ by any non-trivial lattice $\mathbb{B}$ with 0 and 1 in view of ML (modal logic), IL (intuitionistic logic), and so on, we obtain the concept of $\mathbb{B}$-valued functional logical spaces (cf. Remark 3.1). The above process has been generalized to deal with other logics mentioned below so that each "logical system" with a "truth" yields a logical space [5].

A **deduction system** on a set A is a pair (R, D) of a subset D of A and a relation R between $\mathsf{A}^* = \bigcup_{n=0}^{\infty} \mathsf{A}^n$ and A. Deduction systems also naturally occur in logic. Let $(\mathsf{A}, \mathcal{F})$ be one of the logical spaces in the above examples. Then we usually think of sound inference rules on A such as $\dfrac{x \quad x \Rightarrow y}{y}$ or $\dfrac{x \quad y}{x \wedge y}$. Let $\wp$ and $\&$ denote them. Then they are regarded as relations between A^* and A, and so is their union $\wp \cup \&$. Thus, any subset D of A yields deduction systems (R, D) for $\mathsf{R} = \wp, \&$, and $\wp \cup \&$, although D is also usually assumed to be sound.

This paper has two purposes. The one is to study the completeness of deduction systems and subsidiarily of sets with respect to logical spaces abstractly and systematically to the extent described below with illustrations by PL and FPL. The other is to give a theoretical basis for the papers [4] [10] [13] by the author and his collaborators Mizumura and Takaoka who will study MPCL (monophasic case logic) and PPCL (polyphasic case logic) designed for rather a developing branch of MP (mathematical psychology) enjoying interesting problems. They will serve as motivations and examples of applications of the theory in this paper. Thus, the abstract theory of the logical spaces applies to classical PL, FPL, non-classical ML, IL, newborn MPCL, PPCL, and hopefully to other logics. It should be noted that this paper together with [4], [5] and others abridges an aspect of an electronic publication [6] which contains updates on progress in MP and has served as a text at the Graduate School Math. Sci. Univ. Tokyo for more than a decade.

In the literature, there seems to be a debate on the question "What is a logical system?" It will be fruitless, because answers may vary according to a sense of purpose. For instance as for MP, logic is a tool for constructing and analyzing a mathematical model of human system of cognizing and thinking, the outer worlds which humans cognize and think about, and the relationship between a human and the outer worlds, just as calculus is a tool for mechanics. The language A of the logic is a model of the human system of cognition, and semantics for A consists of models (W, Φ) of pairs of an outer world and a relationship between a human and the world. The pair $\big(\mathsf{A}, \{(W, \Phi)\}\big)$ deserves the name of logical system, while deduction systems on A are so named because they are models of the human system of thinking. Adequacy of the two systems is examined solely by observation of human's thought through natural languages, just as Newton's laws are solely examined by observation of motion, neither by mathematician's aesthetic sense nor by engineer's sense of efficiency. On the other hand, the abstract theory of logical spaces is a framework for practical logics, and as such could have been polished by sense of beauty and efficiency, and generalized by wish for a wider use.

The body of this paper begins at §4 and ends at §8. Among the preliminaries collected in §2, however, **quasi-finitarity** (Definition 2.2) is a key concept and leads to Theorem 6.10, which shows how all the **$\mathcal{B}$-theories** (§6) of a logical space $(\mathsf{A}, \mathcal{B})$ are generated by $\mathcal{B}$ and in turn leads to a classification of the logical spaces into three **classes** (Definition 6.3) and also to results on **$\mathcal{B}$-models** (Definition 8.2) of **$\mathcal{B}$-consistent** (Definition 8.1) subsets of A. Also among the preliminaries collected in §3, results in §3.1–§3.2 are regarded as ones on the **$\mathcal{F}$-validity relation** (6.1) of a **Boolean logical space** $(\mathsf{A}, \mathcal{F})$ (Definition 6.1). All of PL, FPL, MPCL, and PPCL yield Boolean logical spaces (Example 6.1). PL and FPL belong to Class 2, while MPCL belongs to Class 2 or 3 according to a parameter (Example 8.2, [13]), and so probably will PPCL. Theorem 7.6 shows that a functional logical space belongs to Class 2 iff every **complete** deduction system on it is **extra-complete** but not **super-complete** (Definition 7.1). Theorem 8.9 together with Example 8.1 shows that a Boolean logical space $(\mathsf{A}, \mathcal{F})$ belongs to Class 1 or 2 iff every $\mathcal{F}$-consistent subset has an $\mathcal{F}$-model. The final §9 includes an answer to a question raised by a referee and suggests an alternative approach to logical spaces.

This paper has a few more key concepts and results. Among them is Theorem 7.12, which gives a practically necessary and sufficient general program (Remark 7.3) for finding a complete deduction system (R, D) with respect to a functional logical space $(\mathsf{A}, \mathcal{F})$. The program reads as follows.

1st step: Find a **characteristic law** (Definition 7.2) $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ of $(\mathsf{A}, \mathcal{F})$.
2nd step: Find an **$\mathcal{F}$-sound** (Definition 7.1) deduction system (R, D) on A whose **deduction relation** (5.2) satisfies the characteristic law $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$.

This program is not quite new. The 1st step amounts to finding a Gentzen style deduction system for sequents, while the 2nd step amounts to converting it into a Hilbert style deduction system for sentences. There already exist several methods for the 1st step, although the two steps are sometimes only implicitly separated. The author does not intend to present a new method within the theory of logical spaces. He and Mizumura [4] have used **resolution trees** for MPCL, and so will Mizumura [10] for PPCL. Takaoka [13] will use Dedekind **cuts** (§7.5) for MPCL. The author learned both methods from Ono [11], although Ono does not use the term "cut." The latter method is also implicit in Henkin's proof [7] of Gödel's completeness theorem [3]. Theorem 7.13 is its theoretical basis and unites the 1st step with the problem of how to construct $\mathcal{F}$-models of $\mathcal{F}$-consistent sets (Remark 7.5). The 2nd step is illustrated in §5.3 by Boolean logical spaces.

A referee has pointed out that there are several theories similar to that of logical spaces, for instance the theory of **abstract logics** of Brown and Suszko [1], which is said to be the most notable of the similar studies. The surveys [2] [8] by Font, Jansana, and Pigozzi are probably good sources of information on more recent such studies.

Suszko writes "The class of abstract logics $\langle \mathcal{A}, \mathcal{C} \rangle$, with algebras $\mathcal{A}$ of a fixed similarity type, constitute a category when supplied with suitably defined morphisms. Here, the analogy with general topology is the guiding idea so that the morphisms of abstract logics are defined as 'continuous' homomorphisms. It is natural to continue the said analogy and, to introduce projective and inductive generation of abstract logics as well as the notion of a dual space. Thus, we construct a general framework of the theory of abstract logics."

Brown and Suszko write "The work of Lindenbaum, Henkin, Rasiowa, and Sikorski show in many cases of interest that an adequate semantics in the sense of completeness theorems can be obtained from the language. For the propositional calculi this is usually done by treating the language as an abstract algebra defining an appropriate congruence and factoring the algebra. The models are then defined as these quotient algebras with a distinguished subset. Such structures have been termed logical matrices in the literature. Usually the quotient algebra is a lattice and the distinguished subset consists of only one element, the unit element in the lattice. An excellent summary of this algebraic approach to semantics is in 'The Mathematics of Metamathematics' by Rasiowa and Sikorski. We show that this construction is a special case of projective generation of abstract logics."

These citations and the above description of the theory of logical spaces clearly show the characteristic of the theory of abstract logics and the difference between the two theories. The author neither pursues the analogy with general topology nor aims at the algebraic construction of models. He and his collaborators also use the language in proving their completeness theorems or constructing models for MPCL and PPCL of interest to them and for PL, FPL, ML, IL, and so on as side examples to them. However, they never factor the language by any congruence. Instead, they use resolution trees and Dedekind cuts, and it seems needless to connect them with algebras. The author was formerly a group theorist and understands importance of quotient algebras. However, after converting from groups to MP, he soon realized that quotients are irrelevant to MP, and thus has been led to the theory of logical spaces as described above.

This paper is almost self-contained and relies only on well-known and basic concepts about sets, orders, and algebras related to lattices. All algebras and operations in this paper are total. Our notation and terminology about them will be standard except that we denote the set of the finite subsets of a set X by $\mathcal{P}'\mathsf{X}$ and that we denote the set of the mappings of a set Y into a set Z by $\mathsf{Y} \to \mathsf{Z}$ instead of Z^{Y}. Thus $\mathsf{f} \in \mathsf{Y} \to \mathsf{Z}$ means $\mathsf{f} : \mathsf{Y} \to \mathsf{Z}$.

Some numbered remarks in this paper include helpful comments, motivations, and interesting examples. Some remarks and even examples are also theorems without proofs. Conversely, some theorems are remarks with proofs. In this sense, there are no big differences between theorems, remarks, and examples.

2. Finitarities and closures

The preliminaries collected in this section will be used mainly in §6.3, §8, the proofs of Theorems 3.28 and 4.4.

2.1. Covers and finitarities

Definition 2.1. Let A be a set, $\mathcal{B}$ be a subset of $\mathcal{P}\mathsf{A}$, and X be a subset of A. Then X is said to be **extra-covered** by $\mathcal{B}$, if for each $\mathsf{Y} \in \mathcal{P}'\mathsf{X}$ there exists an element $\mathsf{B} \in \mathcal{B}$ such that $\mathsf{Y} \subseteq \mathsf{B}$. Also, X is said to be **super-covered** by $\mathcal{B}$, if for each $\mathsf{Y} \in \mathcal{P}'\mathsf{X}$ there exists an element $\mathsf{B} \in \mathcal{B}$ such that $\mathsf{Y} \subseteq \mathsf{B} \subseteq \mathsf{X}$. Also, X is said to be **ultra-covered** by $\mathcal{B}$, if $\mathcal{P}'\mathsf{X} \subseteq \mathcal{B}$.

As an immediate consequence of Definition 2.1, we have that if X is ultra-covered by $\mathcal{B}$, then X is super-covered by $\mathcal{B}$. Also, if X is super-covered by $\mathcal{B}$, then X is extra-covered by $\mathcal{B}$. Also, if X is extra-covered by $\mathcal{B}$, then X is covered by $\mathcal{B}$ in the usual sense that $\mathsf{X} \subseteq \bigcup_{\mathsf{B}\in\mathcal{B}} \mathsf{B}$ holds. Also, if $\mathsf{X} \in \mathcal{B}$, then X is super-covered by $\mathcal{B}$. Furthermore, if $\mathcal{B}$ is **downward** in the sense that $\mathcal{P}\mathsf{B} \subseteq \mathcal{B}$ for each $\mathsf{B} \in \mathcal{B}$, then the concepts of extra-cover, super-cover, and ultra-cover by $\mathcal{B}$ are identical.

Definition 2.2. Let A be a set and $\mathcal{B}$ be a subset of $\mathcal{P}\mathsf{A}$. Then $\mathcal{B}$ is said to be **finitary**, if every subset of A which is extra-covered by $\mathcal{B}$ belongs to $\mathcal{B}$. Also, $\mathcal{B}$ is said to be **quasi-finitary**, if every subset of A which is super-covered by $\mathcal{B}$ belongs to $\mathcal{B}$ (Matsuda [9]). Also, $\mathcal{B}$ is said to be **semi-finitary**, if every subset of A which is ultra-covered by $\mathcal{B}$ belongs to $\mathcal{B}$.

Definition 2.2 and remarks about Definition 2.1 immediately show that if $\mathcal{B}$ is finitary then $\mathcal{B}$ is quasi-finitary and that if $\mathcal{B}$ is quasi-finitary then $\mathcal{B}$ is semi-finitary. Furthermore, the following holds, which in particular shows that the above definition of "finitary" is equivalent to the usual one.

Theorem 2.1. Let A be a set and $\mathcal{B}$ be a subset of $\mathcal{P}\mathsf{A}$. Then the following four conditions are equivalent.

(1) $\mathcal{B}$ is finitary.
(2) $\mathcal{B}$ is downward and quasi-finitary.
(3) $\mathcal{B}$ is downward and semi-finitary.
(4) A subset X of A belongs to $\mathcal{B}$ iff X is ultra-covered by $\mathcal{B}$.

Proof. As elements of $\mathcal{B}$ are extra-covered by $\mathcal{B}$, (1) is equivalent to the condition

(5) a subset X of A belongs to $\mathcal{B}$ iff X is extra-covered by $\mathcal{B}$.

Also, each of the conditions (1)–(5) implies that $\mathcal{B}$ is downward, and if $\mathcal{B}$ is downward, then extra-cover, super-cover, and ultra-cover by $\mathcal{B}$ are identical. Therefore, the conditions (1)–(5) are equivalent. □

Theorem 2.2. Let A be a set and $\mathcal{B}$ be a subset of $\mathcal{P}A$. Assume that $\mathcal{B}$ is quasi-finitary. Then the ordered set $(\mathcal{B}, \subseteq)$ is inductive.

Proof. Let $\{B_i \mid i \in I\}$ be a non-empty totally ordered subset of $\mathcal{B}$. Define $X = \bigcup_{i\in I} B_i$. If $Y \in \mathcal{P}'X$, then there exists an index $i \in I$ such that $Y \subseteq B_i$. Since $B_i \in \mathcal{B}$ and $B_i \subseteq X$, X is super-covered by $\mathcal{B}$, hence $X \in \mathcal{B}$. Therefore $(\mathcal{B}, \subseteq)$ is inductive. □

Definition 2.3. Let A, B be sets and φ be a mapping of $\mathcal{P}A$ into $\mathcal{P}B$. Then φ is said to be **finitary**, if $\varphi X = \bigcup_{Y\in\mathcal{P}'X} \varphi Y$ for each $X \in \mathcal{P}A$.

Note that $\mathcal{P}A \to \mathcal{P}B$ is a complete lattice with respect to the order $\subseteq$ defined for each $(\varphi, \psi) \in (\mathcal{P}A \to \mathcal{P}B)^2$ so that $\varphi \subseteq \psi$ iff $\varphi X \subseteq \psi X$ for all $X \in \mathcal{P}A$.

Theorem 2.3. Let A, B be sets. Then the following holds.

(1) If $\varphi \in \mathcal{P}A \to \mathcal{P}B$ is finitary, then φ is increasing.
(2) If $\varphi \in \mathcal{P}A \to \mathcal{P}B$ is finitary and $\psi \in \mathcal{P}A \to \mathcal{P}B$ is increasing, then the subset $\{X \in \mathcal{P}A \mid \varphi X \subseteq \psi X\}$ of $\mathcal{P}A$ is quasi-finitary, and $\varphi \subseteq \psi$ holds iff $\varphi Y \subseteq \psi Y$ for all $Y \in \mathcal{P}'A$.
(3) If $\varphi \in \mathcal{P}A \to \mathcal{P}B$ and $\psi \in \mathcal{P}B \to \mathcal{P}C$ are finitary for a set C, then the composite mapping $\psi \cdot \varphi \in \mathcal{P}A \to \mathcal{P}C$ is finitary.
(4) If $(\varphi_i)_{i\in I}$ is a family of finitary mappings $\varphi_i \in \mathcal{P}A \to \mathcal{P}B$, then their supremum $\bigcup_{i\in I} \varphi_i$ in $\mathcal{P}A \to \mathcal{P}B$ is also finitary.
(5) If $\varphi \in \mathcal{P}A \to \mathcal{P}B$ is finitary and $D \subseteq A$, then the mapping $X \mapsto \varphi(X \cup D)$ is also finitary.

Proof. The proofs of (1) and (4) are left to the reader.

(2) Suppose $X \in \mathcal{P}A$ is super-covered by $\mathcal{B} = \{B \in \mathcal{P}A \mid \varphi B \subseteq \psi B\}$. Then for each $Y \in \mathcal{P}'X$, there exists a set $B \in \mathcal{B}$ such that $Y \subseteq B \subseteq X$, so $\varphi Y \subseteq \varphi B \subseteq \psi B \subseteq \psi X$. Hence $\varphi X \subseteq \psi X$, that is, $X \in \mathcal{B}$. Therefore $\mathcal{B}$ is quasi-finitary. If $\varphi Y \subseteq \psi Y$ for all $Y \in \mathcal{P}'A$, every member of $\mathcal{P}A$ is ultra-covered by $\mathcal{B}$, hence $\mathcal{P}A \subseteq \mathcal{B}$, which implies $\varphi \subseteq \psi$.

(3) Let $X \in \mathcal{P}A$ and $Z \in \mathcal{P}'(\varphi X)$. Then $Z \subseteq \varphi X = \bigcup_{Y\in\mathcal{P}'X} \varphi Y$, and so for each element $z \in Z$, there exists a set $Y_z \in \mathcal{P}'X$ such that $z \in \varphi Y_z$. Define $Y_Z = \bigcup_{z\in Z} Y_z$. Then $Y_Z \in \mathcal{P}'X$ and $Z \subseteq \varphi(Y_Z)$, hence $\psi Z \subseteq \psi\bigl(\varphi(Y_Z)\bigr)$. Therefore, $\psi(\varphi X) = \bigcup_{Z\in\mathcal{P}'(\varphi X)} \psi Z \subseteq \bigcup_{Z\in\mathcal{P}'(\varphi X)} \psi\bigl(\varphi(Y_Z)\bigr) \subseteq \bigcup_{Y\in\mathcal{P}'X} \psi(\varphi Y)$. Since $\psi \cdot \varphi$ is increasing, we conclude that $\psi \cdot \varphi$ is finitary.

(5) Let δ be the constant mapping $X \mapsto D$ on $\mathcal{P}A$. Then, since $\varphi, \mathrm{id}_{\mathcal{P}A}$, and δ are finitary, so is $\varphi \cdot (\mathrm{id}_{\mathcal{P}A} \cup \delta)$, which is equal to $X \mapsto \varphi(X \cup D)$. □

2.2. Closure operators

A **closure operator** on an ordered set $(A, \leq)$ is a mapping $\varphi \in A \to A$ which satisfies the following three conditions.

(1) φ is **extensive** in the sense that $x \leq \varphi x$ for all $x \in A$.
(2) φ is **idempotent** in the sense that $\varphi(\varphi x) = \varphi x$ for all $x \in A$.
(3) φ is increasing in the usual sense that if $x \leq y$ then $\varphi x \leq \varphi y$.

The set $\{x \in A \mid \varphi x = x\} = \{x \in A \mid \varphi x \leq x\} = \varphi A$ will be called the **fixture domain** of φ. Note that the order $\leq$ on A induces the order $\leq$ on $A \to A$ defined for each $(\varphi, \psi) \in (A \to A)^2$ so that $\varphi \leq \psi$ iff $\varphi x \leq \psi x$ for all $x \in A$.

Theorem 2.4. Let $(A, \leq)$ be an ordered set. Then the following holds.

(1) If φ is a closure operator on $(A, \leq)$ and B is the fixture domain of φ, then $\varphi x = \min\{y \in B \mid x \leq y\}$.
(2) If B is a subset of A such that there exists $\min\{y \in B \mid x \leq y\}$ for each element $x \in A$, then the mapping $x \mapsto \min\{y \in B \mid x \leq y\}$ is a closure operator on $(A, \leq)$ and its fixture domain is equal to B.
(3) Two closure operators φ, ψ on $(A, \leq)$ satisfy $\varphi \leq \psi$ iff the fixture domain of ψ is contained in that of φ.

Proof is left to the reader. This theorem will be used without notice.

Theorem 2.5. Let $(A, \leq)$ be a complete lattice and, for each subset B of A, let $B^\cap$ denote the set $\{\inf X \mid X \subseteq B\}$. Note that $\max A = \inf \emptyset \in B^\cap$. Let us say that B is **$\cap$-closed** in $(A, \leq)$ if $B^\cap = B$. Then $B^\cap$ is the smallest of the $\cap$-closed subsets of A which contain B (for the reason, we call $B^\cap$ the **$\cap$-closure** of B in $(A, \leq)$). Therefore, the mapping $B \mapsto B^\cap$ is a closure operator on $(\mathcal{P}A, \subseteq)$, and its fixture domain is equal to the set of the $\cap$-closed subsets of A.

In particular, if A is a set and $\mathcal{B}$ is a subset of $\mathcal{P}A$, then the following holds for the $\cap$-closure $\mathcal{B}^\cap$ of $\mathcal{B}$ in $(\mathcal{P}A, \subseteq)$.

(1) $\bigcap_{Y \subseteq B \in \mathcal{B}} B = \bigcap_{Y \subseteq X \in \mathcal{B}^\cap} X$ for each $Y \in \mathcal{P}A$.
(2) $\bigcup_{B \in \mathcal{B} - \{A\}} B = \bigcup_{X \in \mathcal{B}^\cap - \{A\}} X$ (note that $A = \max(\mathcal{P}A) = \inf \emptyset \in \mathcal{B}^\cap$).

Proof is left to the reader.

Theorem 2.6. Let A be a set and, for each subset $\mathcal{B}$ of $\mathcal{P}A$, let $\overline{\mathcal{B}}$ be the set of the subsets of A which are super-covered by $\mathcal{B}$. Then the mappping $\mathcal{B} \mapsto \overline{\mathcal{B}}$ is a closure operator on $(\mathcal{P}(\mathcal{P}A), \subseteq)$ and its fixture domain is equal to the set of the quasi-finitary subsets of $\mathcal{P}A$. Therefore, $\overline{\mathcal{B}}$ is the smallest of the quasi-finitary subsets of $\mathcal{P}A$ which contain $\mathcal{B}$ (for the reason, we call $\overline{\mathcal{B}}$ the **quasi-finitary closure** of $\mathcal{B}$). Furthermore the following holds.

(1) $\bigcap_{Y \subseteq B \in \mathcal{B}} B = \bigcap_{Y \subseteq X \in \overline{\mathcal{B}}} X$ for each $Y \in \mathcal{P}'A$.
(2) $\bigcup_{B \in \mathcal{B} - \{A\}} B = \bigcup_{X \in \overline{\mathcal{B}} - \{A\}} X$.
(3) $\overline{\mathcal{B}} - \mathcal{B}$ consists of infinite sets.

Proof. Let $\mathcal{B} \subseteq \mathcal{P}A$. Then the sets in $\mathcal{B}$ are super-covered by $\mathcal{B}$, hence $\mathcal{B} \subseteq \overline{\mathcal{B}}$. If $\mathcal{B} \subseteq \mathcal{B}' \subseteq \mathcal{P}A$, then the sets in $\mathcal{P}A$ super-covered by $\mathcal{B}$ are also super-covered by $\mathcal{B}'$, hence $\overline{\mathcal{B}} \subseteq \overline{\mathcal{B}'}$. Suppose $X \in \mathcal{P}A$ is super-covered by $\overline{\mathcal{B}}$. If $Y \in \mathcal{P}'X$, then there exists a set $B' \in \overline{\mathcal{B}}$ such that $Y \subseteq B' \subseteq X$, and so there exists a set $B \in \mathcal{B}$ such that $Y \subseteq B \subseteq B' \subseteq X$. Thus X is super-covered by $\mathcal{B}$. We have shown $\overline{\overline{\mathcal{B}}} \subseteq \overline{\mathcal{B}}$. Therefore the mappping $\mathcal{B} \mapsto \overline{\mathcal{B}}$ is a closure operator. By Definition 2.2, $\overline{\mathcal{B}} = \mathcal{B}$ iff $\mathcal{B}$ is quasi-finitary. Suppose $Y \in \mathcal{P}'A$ and $Y \subseteq X \in \overline{\mathcal{B}}$. Then there exists a set $B \in \mathcal{B}$ such that $Y \subseteq B \subseteq X$. Therefore (1) holds. If $x \in X \in \overline{\mathcal{B}} - \{A\}$, then there exists a set $B \in \mathcal{B} - \{A\}$ such that $x \in B \subseteq X$. Therefore (2) holds. If X is a finite set in $\overline{\mathcal{B}}$, then since $X \in \mathcal{P}'X$, there exists a set $B \in \mathcal{B}$ such that $X \subseteq B \subseteq X$, hence $X = B \in \mathcal{B}$. Thus (3) holds. □

Theorem 2.7. Let A be a set. Then the mapping $\mathcal{B} \mapsto \overline{\mathcal{B}^{\cap}}$ of $\mathcal{P}(\mathcal{P}A)$ into itself is a closure operator on $(\mathcal{P}(\mathcal{P}A), \subseteq)$, and its fixture domain is equal to the set of the subsets of $\mathcal{P}A$ which are $\cap$-closed in $(\mathcal{P}A, \subseteq)$ and quasi-finitary. Therefore, $\overline{\mathcal{B}^{\cap}}$ is the smallest of the subsets of $\mathcal{P}A$ which contain $\mathcal{B}$ and are $\cap$-closed in $(\mathcal{P}A, \subseteq)$ and quasi-finitary (for the reason, we call $\overline{\mathcal{B}^{\cap}}$ the **quasi-finitary $\cap$-closure** of $\mathcal{B}$).

Proof. We only need to show that if $\mathcal{B}$ is a $\cap$-closed subset of $\mathcal{P}A$ then so is $\overline{\mathcal{B}}$. Let $(X_i)_{i \in I}$ be a family of sets in $\overline{\mathcal{B}}$ and define $X = \bigcap_{i \in I} X_i$. If $Y \in \mathcal{P}'X$, then for each $i \in I$, there exists a set $B_i \in \mathcal{B}$ such that $Y \subseteq B_i \subseteq X_i$. Let $B = \bigcap_{i \in I} B_i$. Then $Y \subseteq B \subseteq X$, and $B \in \mathcal{B}$ because $\mathcal{B}$ is $\cap$-closed. Therefore $X \in \overline{\mathcal{B}}$ as desired. □

Theorem 2.8. Let $(A, \leq)$ be an ordered set. For each element $a \in A$, let $(\leftarrow a]$ denote the downward interval $\{x \in A \mid x \leq a\}$. Also, let us say that a subset B of A is **downward** if $(\leftarrow b] \subseteq B$ for all $b \in B$. Furthermore, for each subset B of A, define $\overleftarrow{B} = \bigcup_{b \in B} (\leftarrow b]$. Then the mapping $B \mapsto \overleftarrow{B}$ is a closure operator on $(\mathcal{P}A, \subseteq)$ and its fixture domain is equal to the set of the downward subsets of A. Therefore, $\overleftarrow{B}$ is the smallest of the downward subsets of A which contain B (for the reason, we call $\overleftarrow{B}$ the **downward closure** of B).

Proof is left to the reader.

Theorem 2.9. Let A be a set and $\mathcal{B}$ be a subset of $\mathcal{P}A$. Then the quasi-finitary closure $\overline{\overleftarrow{\mathcal{B}}}$ of the downward closure $\overleftarrow{\mathcal{B}}$ of $\mathcal{B}$ for $(\mathcal{P}A, \subseteq)$ is equal to the set of the subsets of A which are extra-covered by $\mathcal{B}$, and the mapping $\mathcal{B} \mapsto \overline{\overleftarrow{\mathcal{B}}}$ is a closure operator on $(\mathcal{P}(\mathcal{P}A), \subseteq)$ whose fixture domain is equal to the set of the finitary subsets of $\mathcal{P}A$. Therefore, $\overline{\overleftarrow{\mathcal{B}}}$ is the smallest of the finitary subsets of $\mathcal{P}A$ which contain $\mathcal{B}$ (for the reason, we call $\overline{\overleftarrow{\mathcal{B}}}$ the **finitary closure** of $\mathcal{B}$).

Proof. First assume that $X \in \mathcal{P}A$ is extra-covered by $\mathcal{B}$. If $Y \in \mathcal{P}'X$, then there exists a set $B \in \mathcal{B}$ such that $Y \subseteq B$, hence $X \cap B \in \overleftarrow{\mathcal{B}}$ and $Y \subseteq X \cap B \subseteq X$. Thus $X \in \overline{\overleftarrow{\mathcal{B}}}$. Next assume $X \subseteq X' \in \overline{\overleftarrow{\mathcal{B}}}$. If $Y \in \mathcal{P}'X$, then there exists a set $Z \in \overleftarrow{\mathcal{B}}$ such that $Y \subseteq Z \subseteq X'$, and so there exists a set $B \in \mathcal{B}$ such that $Y \subseteq Z \subseteq B$. Therefore

X is extra-covered by $\mathcal{B}$. We have shown that $\overleftarrow{\overleftarrow{\mathcal{B}}}$ is equal to the set of the subsets of A which are extra-covered by $\mathcal{B}$ and also that $\overleftarrow{\overleftarrow{\mathcal{B}}}$ is downward. Therefore, the mapping $\mathcal{B} \mapsto \overleftarrow{\overleftarrow{\mathcal{B}}}$ is a closure operator and its fixture domain is equal to the set of the finitary subsets of $\mathcal{P}\mathsf{A}$. $\square$

Theorem 2.10. Let φ be a closure operator on a complete lattice $(\mathsf{A}, \leq)$ and let B be the fixture domain of φ. Then the following holds.

(1) $\sup_{\mathsf{B}} \mathsf{X} = \varphi(\sup_{\mathsf{A}} \mathsf{X})$ and $\inf_{\mathsf{B}} \mathsf{X} = \varphi(\inf_{\mathsf{A}} \mathsf{X}) = \inf_{\mathsf{A}} \mathsf{X}$ for all sets $\mathsf{X} \subseteq \mathsf{B}$.
(2) $\min \mathsf{B} = \varphi(\min \mathsf{A})$ and $\max \mathsf{B} = \max \mathsf{A}$.
(3) $\varphi(\sup_{\mathsf{A}}(\varphi\mathsf{Y})) = \varphi(\sup_{\mathsf{A}} \mathsf{Y})$ for all sets $\mathsf{Y} \subseteq \mathsf{A}$, where $\varphi\mathsf{Y} = \{\varphi y \mid y \in \mathsf{Y}\}$.

Consequently, $(\mathsf{B}, \leq)$ is also complete and B is $\cap$-closed in A.

Proof is left to the reader.

Theorem 2.11. Let A be a set, φ be a closure operator on $(\mathcal{P}\mathsf{A}, \subseteq)$, $\mathcal{B}$ be the fixture domain of φ, $\mathsf{D} \subseteq \mathsf{A}$, and define $\psi \in \mathcal{P}\mathsf{A} \to \mathcal{P}\mathsf{A}$ by $\psi\mathsf{X} = \varphi(\mathsf{X} \cup \mathsf{D})$. Then ψ is also a closure operator and its fixture domain is equal to $\{\mathsf{Y} \in \mathcal{B} \mid \mathsf{D} \subseteq \mathsf{Y}\}$. If furthermore φ is finitary, so is ψ.

Proof. Let $\mathcal{B}' = \{\mathsf{Y} \in \mathcal{B} \mid \mathsf{D} \subseteq \mathsf{Y}\}$. Then $\varphi(\mathsf{X} \cup \mathsf{D}) = \min\{\mathsf{Y} \in \mathcal{B}' \mid \mathsf{X} \subseteq \mathsf{Y}\}$, hence the former assertion. The latter is a consequence of Theorem 2.3. $\square$

3. Latticed representations and latticed relations

The preliminaries collected in this section will be used mainly in §5.2, §5.3, §6.2, §7.3, the proofs of Theorems 4.5, 5.2, 6.3, and 8.7.

Throughout this section, we let A be a set, and denote by A^* the set of all formal products $x_1 \cdots x_n$ of elements $x_1, \dots, x_n$ of A of arbitrary finite length $n \geq 0$. In other words, A^* is the free monoid on A. We denote elements of A by $x, y, \dots$, while elements of A^* by $\alpha, \beta, \dots$, both with or without numerical subscripts. This **alphabetical convention** will be used throughout the remainder of this paper. In particular, the element of A^* of length 0 or the identity element of the monoid A^* will be denoted by ε. When $\alpha = x_1 \cdots x_n \in \mathsf{A}^*$, we denote the subset $\{x_1, \dots, x_n\}$ of A also by α, where if $n = 0$, then $\alpha = \varepsilon$ and $\{x_1, \dots, x_n\} = \emptyset$. This **sequence convention** will also be used throughout the remainder of this paper.

Let S, T be sets. Then the relations between S and T may be regarded as the subsets of $\mathsf{S} \times \mathsf{T}$, and so we may discuss the order $\mathsf{R} \subseteq \mathsf{Q}$ of relations R, Q between S and T, and if $\mathsf{R} \subseteq \mathsf{Q}$, we may say that R is contained in Q or that Q contains R. In particular, the term "largest" in Definition 3.3, Theorems 3.20, 6.1, and so on means "largest with respect to the order $\subseteq$," and similarly for the term "smallest."

3.1. Validity relations of latticed representations

Throughout this subsection, we let $\mathbb{B}$ be a lattice with respect to the order $\leq$, meet $\wedge$, and join $\vee$, and with the smallest element 0 and the largest element 1. Then a **latticed representation** of A on $\mathbb{B}$ is simply a mapping f of A into $\mathbb{B}$.

Remark 3.1. If f is a latticed representation of a non-empty set A on a non-trivial lattice, then $(A, \{f\})$ is a functional logical space. Conversely, if $(A, \mathcal{F})$ is a $\mathbb{B}$-valued functional logical space with $\mathcal{F} \neq \emptyset$, $(A, \mathcal{F})$ is "equivalent" to a functional logical space $(A, \{\varphi\})$ made of a single latticed representation φ of A on the power lattice $\mathbb{B}^{\mathcal{F}} = \mathcal{F} \to \mathbb{B}$ defined by $(\varphi x)f = fx$ for each $x \in A$ and $f \in \mathcal{F}$. *This is a reason why latticed representations are relevant to the theory of logical spaces.* Also, extension of the concept of $\mathbb{T}$-valued functional logical spaces to that of lattice-valued functional logical spaces does not only mean extension of truth values, but also means bundling of truth functions.

For the latticed representation $(A, \mathbb{B}, f)$, we define the relation $\preccurlyeq_f$ on A^* by

$$\alpha \preccurlyeq_f \beta \iff \inf f\alpha \leq \sup f\beta \tag{3.1}$$

using the sequence convention. We call $\preccurlyeq_f$ the **f-validity relation**. If $A \neq \emptyset$ and $\mathbb{B}$ is non-trivial, then $\preccurlyeq_f$ is equal to the $\{f\}$-validity relation to be defined by (6.1) for the $\mathbb{B}$-valued functional logical space $(A, \{f\})$ (cf. Remark 6.1).

In this subsection, we study properties of the f-validity relation $\preccurlyeq_f$ under various algebraic additional conditions on the latticed representation $(A, \mathbb{B}, f)$. It includes the study of $\mathbb{B}$ itself because $(\mathbb{B}, \mathbb{B}, \mathrm{id}_{\mathbb{B}})$ is a latticed representation.

Theorem 3.1. The f-validity relation $\preccurlyeq_f$ satisfies the following four laws:

$$x \preccurlyeq_f x, \qquad \text{(repetition law)}$$

$$\left.\begin{array}{l} \alpha \preccurlyeq_f \beta \implies x\alpha \preccurlyeq_f \beta, \\ \alpha \succcurlyeq_f \beta \implies x\alpha \succcurlyeq_f \beta, \end{array}\right\} \qquad \text{(weakening law)}$$

$$\left.\begin{array}{l} xx\alpha \preccurlyeq_f \beta \implies x\alpha \preccurlyeq_f \beta, \\ xx\alpha \succcurlyeq_f \beta \implies x\alpha \succcurlyeq_f \beta, \end{array}\right\} \qquad \text{(contraction law)}$$

$$\left.\begin{array}{l} \alpha xy\beta \preccurlyeq_f \gamma \implies \alpha yx\beta \preccurlyeq_f \gamma, \\ \alpha xy\beta \succcurlyeq_f \gamma \implies \alpha yx\beta \succcurlyeq_f \gamma. \end{array}\right\} \qquad \text{(exchange law)}$$

Proof is left to the reader.

Remark 3.2. The repetition law is related to but different from the reflexivity law $\alpha \preccurlyeq_f \alpha$. The $\preccurlyeq_f$ is reflexive iff $\mathbb{B}$ is trivial, because $\varepsilon \preccurlyeq_f \varepsilon$ iff $1 \leq 0$.

Theorem 3.2. The f-validity relation $\preccurlyeq_f$ satisfies the following law:

$$\left.\begin{array}{l} \alpha \preccurlyeq_f x,\ x\beta \preccurlyeq_f \delta \implies \alpha\beta \preccurlyeq_f \delta, \\ \alpha \succcurlyeq_f x,\ x\beta \succcurlyeq_f \delta \implies \alpha\beta \succcurlyeq_f \delta. \end{array}\right\} \qquad \text{(cut law)}$$

If $\mathbb{B}$ is a distributive lattice, then $\preccurlyeq_f$ satisfies the following law:

$$\left.\begin{array}{l} \alpha \preccurlyeq_f x\gamma, \\ x\beta \preccurlyeq_f \delta \end{array}\right\} \implies \alpha\beta \preccurlyeq_f \delta\gamma. \qquad \text{(strong cut law)}$$

Proof. In order to prove the strong cut law, let $a = \inf f\alpha$, $b = \inf f\beta$, $c = \sup f\gamma$, $d = \sup f\delta$, and $e = fx$. Then the premise implies that $a \leq e \vee c$ and $e \wedge b \leq d$ holds. Therefore, if $\mathbb{B}$ is distributive, then $a \wedge b \leq (e \vee c) \wedge b \leq (e \wedge b) \vee c \leq d \vee c$. Therefore $\inf f(\alpha\beta) = a \wedge b \leq d \vee c = \sup f(\delta\gamma)$, hence $\alpha\beta \preccurlyeq_f \delta\gamma$. If $\gamma = \varepsilon$

or $\beta = \varepsilon$, then $c = 0$ or $b = 1$, and so the above reasoning works without the distributive law. Thus the cut law holds with no additional conditions. $\square$

Definition 3.1. If a relation on A^* satisfies the repetition law, weakening law, contraction law, exchange law, and cut law, we say that it is a **latticed relation**. Also, if a relation on A^* satisfies the repetition law, weakening law, contraction law, exchange law, and strong cut law, we say that it is a **strongly latticed relation**.

Thus, the f-validity relation $\preccurlyeq_f$ is a latticed relation, and if $\mathbb{B}$ is distributive, then $\preccurlyeq_f$ is a strongly latticed relation.

Theorem 3.3. The image fA of f contains 0 and 1 iff the f-validity relation $\preccurlyeq_f$ satisfies the following **end laws**:

$$\text{there exists an element } x \in A \text{ such that } x \preccurlyeq_f \varepsilon, \qquad \text{(lower end law)}$$

$$\text{there exists an element } x \in A \text{ such that } x \succcurlyeq_f \varepsilon. \qquad \text{(upper end law)}$$

Proof. This is because an element $x \in A$ satisfies $fx = 0$ iff $x \preccurlyeq_f \varepsilon$, while x satisfies $fx = 1$ iff $x \succcurlyeq_f \varepsilon$. $\square$

Theorem 3.4. According as $0 = \inf fA$ or $1 = \sup fA$, the f-validity relation $\preccurlyeq_f$ satisfies the following lower quasi-end law or the upper quasi-end law (the union of these laws will be called the **quasi-end laws**):

$$\alpha \preccurlyeq_f \varepsilon \iff \alpha \preccurlyeq_f y \text{ for every element } y \in A, \qquad \text{(lower quasi-end law)}$$

$$\alpha \succcurlyeq_f \varepsilon \iff \alpha \succcurlyeq_f y \text{ for every element } y \in A. \qquad \text{(upper quasi-end law)}$$

Proof. Assume $0 = \inf fA$. If an element $\alpha \in A^*$ satisfies $\alpha \preccurlyeq_f y$ for every element $y \in A$, then $\inf f\alpha \leq fy$ for every element $y \in A$, and so $\inf f\alpha \leq \inf fA = 0$, hence $\alpha \preccurlyeq_f \varepsilon$. Conversely if $\alpha \preccurlyeq_f \varepsilon$, then $\alpha \preccurlyeq_f y$ for every element $y \in A$ by the weakening law. The rest of the proof is similar. $\square$

Remark 3.3. Under the weakening law and cut law, the lower and upper end laws respectively imply the lower and upper quasi-end laws.

Theorem 3.5. Let $x \wedge y$ and $x \vee y$ be binary operations on A. Then f is a $\{\wedge, \vee\}$-homomorphism iff the f-validity relation $\preccurlyeq_f$ satisfies the following **junction laws**:

$$x \wedge y \preccurlyeq_f x, \qquad x \wedge y \preccurlyeq_f y, \qquad xy \preccurlyeq_f x \wedge y, \qquad \text{(conjunction law)}$$

$$x \vee y \succcurlyeq_f x, \qquad x \vee y \succcurlyeq_f y, \qquad xy \succcurlyeq_f x \vee y. \qquad \text{(disjunction law)}$$

Proof. Assume that f is a $\{\wedge\}$-homomorphism. Then, for each $(x, y) \in A \times A$, we have $f(x \wedge y) = fx \wedge fy$, hence $f(x \wedge y) \leq fx$, $f(x \wedge y) \leq fy$, and $fx \wedge fy \leq f(x \wedge y)$. These inequalities show that $\preccurlyeq_f$ satisfies the conjunction law. Conversely, assume that $\preccurlyeq_f$ satisfies the conjunction law. Then for each $(x, y) \in A \times A$, we have $f(x \wedge y) \leq fx$, $f(x \wedge y) \leq fy$, and $fx \wedge fy \leq f(x \wedge y)$, hence $f(x \wedge y) = fx \wedge fy$. Thus f is a $\{\wedge\}$-homomorphism. The proof for $\vee$ is similar. $\square$

Theorem 3.6. The image $\mathsf{f}\mathsf{A}$ of f is a sublattice of $\mathbb{B}$ iff the f-validity relation $\preccurlyeq_{\mathsf{f}}$ satisfies the following **quasi-junction laws**:

$$\text{for each } (x, y) \in A \times A, \text{ there exists an element } z \in A \text{ such that}$$
$$z \preccurlyeq_f x,\ z \preccurlyeq_f y, \text{ and } xy \preccurlyeq_f z, \qquad \text{(quasi-conjunction law)}$$
$$\text{for each } (x, y) \in A \times A, \text{ there exists an element } z \in A \text{ such that}$$
$$z \succcurlyeq_f x,\ z \succcurlyeq_f y, \text{ and } xy \succcurlyeq_f z. \qquad \text{(quasi-disjunction law)}$$

Proof. Let $(x, y) \in A \times A$. If fA is closed under the operation $\wedge$, then there exists an element $z \in A$ such that $fx \wedge fy = fz$, hence $fz \leq fx,\ fz \leq fy$, and $fx \wedge fy \leq fz$, which imply $z \preccurlyeq_f x,\ z \preccurlyeq_f y$, and $xy \preccurlyeq_f z$ respectively. Conversely, if there exists an element $z \in A$ which satisfies $z \preccurlyeq_f x,\ z \preccurlyeq_f y$, and $xy \preccurlyeq_f z$, then $fz \leq fx,\ fz \leq fy$, and $fx \wedge fy \leq fz$, hence $fx \wedge fy = fz \in fA$. The rest of the proof is similar. $\square$

Theorem 3.7. Assume that $\mathbb{B}$ is a Boolean lattice, and let $\Diamond$ denote the complement on $\mathbb{B}$. Furthermore let $x^{\Diamond}$ be a unary operation on A. Then f is a $\{\Diamond\}$-homomorphism iff the f-validity relation $\preccurlyeq_f$ satisfies the following **negation laws**:

$$xx^{\Diamond} \preccurlyeq_f \varepsilon, \qquad \text{(lower negation law)}$$
$$xx^{\Diamond} \succcurlyeq_f \varepsilon. \qquad \text{(upper negation law)}$$

Proof. If f is a $\{\Diamond\}$-homomorphism, then we have $fx \wedge f(x^{\Diamond}) = fx \wedge (fx)^{\Diamond} = 0$ and $fx \vee f(x^{\Diamond}) = fx \vee (fx)^{\Diamond} = 1$ for every element $x \in A$, and so $\preccurlyeq_f$ satisfies the negation laws. Conversely, assume that $\preccurlyeq_f$ satisfies the negation laws. Then for every element $x \in A$, we have $fx \wedge f(x^{\Diamond}) = 0$ and $fx \vee f(x^{\Diamond}) = 1$, hence $f(x^{\Diamond}) = (fx)^{\Diamond}$ by the uniqueness of the complement. Thus f is a $\{\Diamond\}$-homomorphism. $\square$

Remark 3.4. In dealing with several lattices simultaneously, we wish to use different symbols for meets and joins in different lattices, for instance, $\cap$ and $\cup$ for a lattice A, $\wedge$ and $\vee$ for a lattice B, $\sqcap$ and $\sqcup$ for a lattice C, and so on. Then, how about complements in Boolean lattices? The best way is to use symbols made of those for meets and joins. For instance, if the lattices $A, B, C, \ldots$ above are Boolean, then use $\circ$ for A, $\Diamond$ for B, $\Box$ for C, and so on. This is the reason why an unusual symbol $\Diamond$ is used for complements or negations in this paper.

Theorem 3.8. Assume that $\mathbb{B}$ is a Boolean lattice, and let $\Diamond$ and $\Rightarrow$ be the complement and implication on $\mathbb{B}$. Furthermore let $x^{\Diamond}$ and $x \Rightarrow y$ be unary and binary operations on A. Then f is a $\{\Diamond, \Rightarrow\}$-isomorphism iff the f-validity relation $\preccurlyeq_f$ satisfies the following **implication laws** in addition to the negation laws:

$$x^{\Diamond} \preccurlyeq_f x \Rightarrow y, \qquad \text{(1st implication law)}$$
$$y \preccurlyeq_f x \Rightarrow y, \qquad \text{(2nd implication law)}$$
$$x \Rightarrow y \preccurlyeq_f x^{\Diamond} y. \qquad \text{(3rd implication law)}$$

Proof. Assume that f is a $\{\Diamond, \Rightarrow\}$-homomorphism. Then $\preccurlyeq_f$ satisfies the negation laws by Theorem 3.7. Furthermore $f(x \Rightarrow y) = fx \Rightarrow fy = (fx)^{\Diamond} \vee fy = f(x^{\Diamond}) \vee fy$, hence inequalities $\mathsf{f}(\mathsf{x}^\Diamond) \le \mathsf{f}(\mathsf{x} \Rightarrow \mathsf{y})$, $\mathsf{fy} \le \mathsf{f}(\mathsf{x} \Rightarrow \mathsf{y})$, and $\mathsf{f}(\mathsf{x} \Rightarrow \mathsf{y}) \le \mathsf{f}(\mathsf{x}^\Diamond) \vee \mathsf{fy}$, which imply that the implication laws are satisfied. Conversely, assume that $\preccurlyeq_\mathsf{f}$ satisfies the negation laws and implication laws. Then f is a $\{\Diamond\}$-homomorphism by Theorem 3.7. Furthermore $(\mathsf{fx})^\Diamond = \mathsf{f}(\mathsf{x}^\Diamond) \le \mathsf{f}(\mathsf{x} \Rightarrow \mathsf{y})$ and $\mathsf{fy} \le \mathsf{f}(\mathsf{x} \Rightarrow \mathsf{y})$ by the 1st and 2nd implication laws, hence $\mathsf{fx} \Rightarrow \mathsf{fy} = (\mathsf{fx})^\Diamond \vee \mathsf{fy} \le \mathsf{f}(\mathsf{x} \Rightarrow \mathsf{y})$. On the other hand, $\mathsf{f}(\mathsf{x} \Rightarrow \mathsf{y}) \le \mathsf{f}(\mathsf{x}^\Diamond) \vee \mathsf{fy} \le (\mathsf{fx})^\Diamond \vee \mathsf{fy} = \mathsf{fx} \Rightarrow \mathsf{fy}$ by the 3rd implication law. Therefore $\mathsf{f}(\mathsf{x} \Rightarrow \mathsf{y}) = \mathsf{fx} \Rightarrow \mathsf{fy}$. Thus f is an $\{\Rightarrow\}$-homomorphism. □

Definition 3.2. Assume that $\mathbb{B}$ is a Boolean lattice, and let $\Diamond$ and $\Rightarrow$ denote the complement and implication on $\mathbb{B}$. Also, assume that $\mathsf{x} \wedge \mathsf{y}$, $\mathsf{x} \vee \mathsf{y}$, $\mathsf{x}^\Diamond$, $\mathsf{x} \Rightarrow \mathsf{y}$ are operations on A, and $\mathsf{f} \in \mathsf{A} \to \mathbb{B}$ is a $\{\wedge, \vee, \Diamond, \Rightarrow\}$-homomorphism. Then we say that f is a **Boolean representation** of A on $\mathbb{B}$ with respect to the operations $\wedge, \vee, \Diamond, \Rightarrow$. Boolean representations on $\mathbb{T} = \{0, 1\}$ are called **binary** representations.

Definition 3.3. Let $\mathsf{x} \wedge \mathsf{y}$, $\mathsf{x} \vee \mathsf{y}$, $\mathsf{x}^\Diamond$, $\mathsf{x} \Rightarrow \mathsf{y}$ be operations on A. Then a relation $\preccurlyeq$ on A^* is called a **Boolean relation** if it is a strongly latticed relation and satisfies the junction laws, negation laws, and implication laws with respect to $\wedge, \vee, \Diamond, \Rightarrow$.

Also, a relation $\preccurlyeq$ on A^* is said to be **weakly Boolean**, if it satisfies the repetition law, weakening law, contraction law, exchange law, and the following four laws with respect to the operations $\wedge, \vee, \Diamond, \Rightarrow$:

$$\left.\begin{array}{l} \mathsf{xy}\alpha \preccurlyeq \beta \implies \mathsf{x} \wedge \mathsf{y}, \alpha \preccurlyeq \beta, \\ \alpha \preccurlyeq \mathsf{x}\beta,\ \alpha \preccurlyeq \mathsf{y}\beta \implies \alpha \preccurlyeq \mathsf{x} \wedge \mathsf{y}, \beta, \end{array}\right\} \qquad \text{(strong conjunction law)}$$

$$\left.\begin{array}{l} \mathsf{x}\alpha \preccurlyeq \beta,\ \mathsf{y}\alpha \preccurlyeq \beta \implies \mathsf{x} \vee \mathsf{y}, \alpha \preccurlyeq \beta, \\ \alpha \preccurlyeq \mathsf{xy}\beta \implies \alpha \preccurlyeq \mathsf{x} \vee \mathsf{y}, \beta, \end{array}\right\} \qquad \text{(strong disjunction law)}$$

$$\left.\begin{array}{l} \alpha \preccurlyeq \mathsf{x}\beta \implies \mathsf{x}^\Diamond\alpha \preccurlyeq \beta, \\ \mathsf{x}\alpha \preccurlyeq \beta \implies \alpha \preccurlyeq \mathsf{x}^\Diamond\beta, \end{array}\right\} \qquad \text{(strong negation law)}$$

$$\left.\begin{array}{l} \alpha \preccurlyeq \mathsf{x}\beta,\ \mathsf{y}\alpha \preccurlyeq \beta \implies \mathsf{x} \Rightarrow \mathsf{y}, \alpha \preccurlyeq \beta, \\ \mathsf{x}\alpha \preccurlyeq \mathsf{y}\beta \implies \alpha \preccurlyeq \mathsf{x} \Rightarrow \mathsf{y}, \beta. \end{array}\right\} \qquad \text{(strong implication law)}$$

The strong conjunction law and strong disjunction law put together will be called the **strong junction laws**. Obviously, the largest relation on A^* is Boolean and weakly Boolean. We call it the **trivial** relation.

Note that weakly Boolean relations need not satisfy the cut law. The relationship between the Boolean relations and the weakly Boolean relations will be clarified by Theorem 3.17.

Theorem 3.9. Assume that $\mathbb{B}$ is a Boolean lattice and f is a Boolean representation with respect to operations $\mathsf{x} \wedge \mathsf{y}$, $\mathsf{x} \vee \mathsf{y}$, $\mathsf{x}^\Diamond$, $\mathsf{x} \Rightarrow \mathsf{y}$ on A. Then the f-validity relation $\preccurlyeq_\mathsf{f}$ is a Boolean relation with respect to the operations.

Proof. This is a synthesis of Theorems 3.1, 3.2, 3.5, and 3.8. □

Theorem 3.10. Assume that $\mathsf{fA} = \mathbb{B}$ and that the f-validity relation $\preccurlyeq_\mathsf{f}$ satisfies either the strong conjunction law with respect to an operation $\mathsf{x} \wedge \mathsf{y}$ on A or the strong disjunction law with respect to an operation $\mathsf{x} \vee \mathsf{y}$ on A. Then $\mathbb{B}$ is a distributive lattice.

Proof. Consider the case where $\preccurlyeq_f$ satisfies the strong conjunction law. Then, since $xy \preccurlyeq_f x$ and $xy \preccurlyeq_f y$ hold by the repetition law, weakening law, and exchange law, it follows that $\preccurlyeq_f$ satisfies the conjunction law. Therefore, f is a $\{\wedge\}$-homomorphism, as shown in the proof of Theorem 3.5. Let $a, b, c, d \in \mathbb{B}$ and assume $d \leq a \vee c,\ d \leq b \vee c$. Then there exist elements $x, y, z, w \in A$ such that $a = fx,\ b = fy,\ c = fz,\ d = fw$, which necessarily satisfy $w \preccurlyeq_f xz,\ w \preccurlyeq_f yz$. Therefore $w \preccurlyeq_f x \wedge y, z$ by the strong conjunction law, hence $fw \leq f(x \wedge y) \vee fz$. Since $f(x \wedge y) = fx \wedge fy$, we conclude that $d \leq (a \wedge b) \vee c$ holds. Therefore $\mathbb{B}$ is distributive. $\square$

Theorem 3.11. Assume that the latticed representation $(A, \mathbb{B}, f)$ satisfies the following condition for each element $\alpha \in A^*$:

$$\left.\begin{aligned} a = \inf f\alpha \implies a = \sup\bigl(fA \cap (\leftarrow a]\bigr), \\ a = \sup f\alpha \implies a = \inf\bigl(fA \cap [a \rightarrow)\bigr), \end{aligned}\right\} \tag{3.2}$$

where $(\leftarrow a] = \{b \in \mathbb{B} \mid b \leq a\}$ and $[a \rightarrow) = \{b \in \mathbb{B} \mid a \leq b\}$. Let $\leq_f$ be the restriction of $\preccurlyeq_f$ to $A \times A$, that is, $x \leq_f y$ iff $fx \leq fy$. Then $x_1 \cdots x_m \preccurlyeq_f y_1 \cdots y_n$ holds iff the following holds for all elements $(x, y) \in A \times A$ (cf. Remark 3.5):

$$x \leq_f x_i\ (i = 1, \ldots, m),\ y_j \leq_f y\ (j = 1, \ldots, n) \implies x \leq_f y. \tag{3.3}$$

Proof. Let $a = \inf\{fx_1, \ldots, fx_m\}$ and $b = \sup\{fy_1, \ldots, fy_n\}$. Then $x \leq_f x_i$ $(i = 1, \ldots, m)$ iff $fx \leq a$, and $y_j \leq_f y$ $(j = 1, \ldots, n)$ iff $b \leq fy$. Therefore, we may argue as follows to complete the proof:

$$\begin{aligned} &(3.3) \text{ holds for all elements } (x, y) \in A \times A \\ &\iff \text{if elements } x, y \in A \text{ satisfy } fx \leq a \text{ and } b \leq fy, \text{ then } fx \leq fy \\ &\iff \text{if elements } c, d \in fA \text{ satisfy } c \leq a \text{ and } b \leq d, \text{ then } c \leq d \\ &\iff \sup\bigl(fA \cap (\leftarrow a]\bigr) \leq \inf\bigl(fA \cap [b \rightarrow)\bigr) \\ &\iff a \leq b \\ &\iff x_1 \cdots x_m \preccurlyeq_f y_1 \cdots y_n. \end{aligned}$$

$\square$

3.2. Analysis of latticed relations

Throughout this subsection, we let A be a set and $\preccurlyeq$ be a relation on A^* which satisfies the repetition law, weakening law, contraction law, and exchange law. Here we search for the laws which are equivalent to the strong cut law, junction laws, negation laws, and implication laws on $\preccurlyeq$.

Theorem 3.12. Let $x \wedge y$ be a binary operation on A, and assume that $\preccurlyeq$ satisfies the cut law. Then the following three laws are equivalent.

($\wedge$1) $x \wedge y \preccurlyeq x,\ x \wedge y \preccurlyeq y$.
($\wedge$2) $xy\alpha \preccurlyeq \beta \implies x \wedge y, \alpha \preccurlyeq \beta$.
($\wedge$3) $\alpha \preccurlyeq x \wedge y, \beta \implies \alpha \preccurlyeq x\beta,\ \alpha \preccurlyeq y\beta$.

Also, the following law ($\wedge$4) and ($\wedge$5) are equivalent and ($\wedge$6) imply them. If $\preccurlyeq$ satisfies the strong cut law, then ($\wedge$6) is equivalent to ($\wedge$4) and to ($\wedge$5).

($\wedge$4) $xy \preccurlyeq x \wedge y$.
($\wedge$5) $x \wedge y, \alpha \preccurlyeq \beta \implies xy\alpha \preccurlyeq \beta$.
($\wedge$6) $\alpha \preccurlyeq x\beta,\ \alpha \preccurlyeq y\beta \implies \alpha \preccurlyeq x \wedge y, \beta$.

Let $x \vee y$ be a binary operation on A, and assume that $\preccurlyeq$ satisfies the cut law. Then the following three laws are equivalent.

($\vee$1) $x \preccurlyeq x \vee y,\ y \preccurlyeq x \vee y$.
($\vee$2) $\alpha \preccurlyeq xy\beta \implies \alpha \preccurlyeq x \vee y, \beta$.
($\vee$3) $x \vee y, \alpha \preccurlyeq \beta \implies x\alpha \preccurlyeq \beta,\ y\alpha \preccurlyeq \beta$.

Also, the following law ($\vee$4) and ($\vee$5) are equivalent and ($\vee$6) imply them. If $\preccurlyeq$ satisfies the strong cut law, then ($\vee$6) is equivalent to ($\vee$4) and to ($\vee$5).

($\vee$4) $x \vee y \preccurlyeq xy$.
($\vee$5) $\alpha \preccurlyeq x \vee y, \beta \implies \alpha \preccurlyeq xy\beta$.
($\vee$6) $x\alpha \preccurlyeq \beta,\ y\alpha \preccurlyeq \beta \implies x \vee y, \alpha \preccurlyeq \beta$.

Proof. The law ($\wedge$2) is derived from ($\wedge$1) by the cut law, exchange law, and contraction law. We have $xy \preccurlyeq x$ and $xy \preccurlyeq y$ by the repetition law, weakening law, and exchange law, and so ($\wedge$2) with $\alpha = \varepsilon$ and $\beta = x$ or y implies ($\wedge$1). The law ($\wedge$3) is derived from ($\wedge$1) by the cut law. We have $x \wedge y \preccurlyeq x \wedge y$ by the repetition law, and so ($\wedge$3) with $\alpha = x \wedge y$ and $\beta = \varepsilon$ implies ($\wedge$1). The law ($\wedge$5) is derived from ($\wedge$4) by the cut law. The law ($\wedge$5) with $\alpha = \varepsilon$ and $\beta = x \wedge y$ implies ($\wedge$4). The law ($\wedge$6) is derived from ($\wedge$4) by the strong cut law, exchange law, and contraction law. The law ($\wedge$6) with $\alpha = xy$ and $\beta = \varepsilon$ implies ($\wedge$4). Since the dual $\succcurlyeq$ of $\preccurlyeq$ is also a latticed relation and dual of the strong cut law is the strong cut law itself, the rest holds by the duality. $\square$

Theorem 3.13. Let $x^\diamond$ be a unary operation on A, and assume that $\preccurlyeq$ satisfies the strong cut law. Then the following four laws are equivalent.

($\diamond$1) $xx^\diamond \preccurlyeq \varepsilon$ (the lower negation law).
($\diamond$2) $\alpha \preccurlyeq x\beta,\ \alpha \preccurlyeq x^\diamond\beta \implies \alpha \preccurlyeq \beta$.
($\diamond$3) $\alpha \preccurlyeq x\beta \implies x^\diamond\alpha \preccurlyeq \beta$.
($\diamond$4) $\alpha \preccurlyeq x^\diamond\beta \implies x\alpha \preccurlyeq \beta$.

Also, the following four laws are equivalent.

($\diamond$5) $\varepsilon \preccurlyeq xx^\diamond$ (the upper negation law).
($\diamond$6) $x\alpha \preccurlyeq \beta,\ x^\diamond\alpha \preccurlyeq \beta \implies \alpha \preccurlyeq \beta$.
($\diamond$7) $x\alpha \preccurlyeq \beta \implies \alpha \preccurlyeq x^\diamond\beta$.
($\diamond$8) $x^\diamond\alpha \preccurlyeq \beta \implies \alpha \preccurlyeq x\beta$.

Proof. The laws ($\diamond$2), ($\diamond$3), and ($\diamond$4) are all derived from ($\diamond$1) by the strong cut law, exchange law, and contraction law. We have $xx^\diamond \preccurlyeq x$ and $xx^\diamond \preccurlyeq x^\diamond$ by the repetition law, weakening law, and exchange law, and so the law ($\diamond$2) with $\alpha = xx^\diamond$ and $\beta = \varepsilon$ implies ($\diamond$1). Similarly, the law ($\diamond$3) with $\alpha = x$ and $\beta = \varepsilon$ together with the exchange law implies ($\diamond$1), and the law ($\diamond$4) with $\alpha = x^\diamond$ and $\beta = \varepsilon$ implies ($\diamond$1). The rest holds by duality. $\square$

Theorem 3.14. Let $x \Rightarrow y$ be a binary operation on A, and assume that $\preccurlyeq$ satisfies the strong cut law and the negation laws with respect to a unary operation $x^\Diamond$ on A. Then the following three laws are equivalent.

($\Rightarrow$1) $x^\Diamond \preccurlyeq x \Rightarrow y,\ y \preccurlyeq x \Rightarrow y$ (the 1st and 2nd implication laws united).

($\Rightarrow$2) $x\alpha \preccurlyeq y\beta \implies \alpha \preccurlyeq x \Rightarrow y, \beta$.

($\Rightarrow$3) $x \Rightarrow y, \alpha \preccurlyeq \beta \implies \alpha \preccurlyeq x\beta,\ y\alpha \preccurlyeq \beta$.

Also, the following four laws are equivalent.

($\Rightarrow$4) $x \Rightarrow y \preccurlyeq x^\Diamond y$ (the 3rd implication law).

($\Rightarrow$5) $\alpha \preccurlyeq x \Rightarrow y, \beta \implies x\alpha \preccurlyeq y\beta$.

($\Rightarrow$6) $\alpha \preccurlyeq x\beta,\ y\alpha \preccurlyeq \beta \implies x \Rightarrow y, \alpha \preccurlyeq \beta$.

($\Rightarrow$7) $x, x \Rightarrow y \preccurlyeq y$ (cut-implication law).

Proof. Assume ($\Rightarrow$1) and $x\alpha \preccurlyeq y\beta$. Then applying the cut law to $x\alpha \preccurlyeq y\beta$ and $y \preccurlyeq x \Rightarrow y$, we have $x\alpha \preccurlyeq x \Rightarrow y, \beta$. Also, applying the weakening law and exchange law to $x^\Diamond \preccurlyeq x \Rightarrow y$, we have $x^\Diamond\alpha \preccurlyeq x \Rightarrow y, \beta$. Hence $\alpha \preccurlyeq x \Rightarrow y, \beta$ by the law ($\Diamond$6). Thus ($\Rightarrow$1) implies ($\Rightarrow$2). Assume ($\Rightarrow$1) and $x \Rightarrow y, \alpha \preccurlyeq \beta$. Then applying the cut law to $x \Rightarrow y, \alpha \preccurlyeq \beta$ and $x^\Diamond \preccurlyeq x \Rightarrow y$, we have $x^\Diamond\alpha \preccurlyeq \beta$, hence $\alpha \preccurlyeq x\beta$ by the law ($\Diamond$8). Also, applying the cut law to $x \Rightarrow y, \alpha \preccurlyeq \beta$ and $y \preccurlyeq x \Rightarrow y$, we have $y\alpha \preccurlyeq \beta$. Thus ($\Rightarrow$1) implies ($\Rightarrow$3). We have $xx^\Diamond \preccurlyeq y$ by the lower negation law and weakening law, and so ($\Rightarrow$2) with $\alpha = x^\Diamond$ and $\beta = \varepsilon$ implies $x^\Diamond \preccurlyeq x \Rightarrow y$. We have $xy \preccurlyeq y$ by the repetition law and weakening law, and so ($\Rightarrow$2) with $\alpha = y$ and $\beta = \varepsilon$ implies $y \preccurlyeq x \Rightarrow y$. Thus ($\Rightarrow$2) implies ($\Rightarrow$1). We have $x \Rightarrow y \preccurlyeq x \Rightarrow y$ by the repetition law, and so ($\Rightarrow$3) with $\alpha = \varepsilon$ and $\beta = x \Rightarrow y$ implies $\varepsilon \preccurlyeq x, x \Rightarrow y$ and $y \preccurlyeq x \Rightarrow y$. Applying the law ($\Diamond$3) to $\varepsilon \preccurlyeq x, x \Rightarrow y$, we have $x^\Diamond \preccurlyeq x \Rightarrow y$. Thus ($\Rightarrow$3) implies ($\Rightarrow$1).

The laws ($\Diamond$4) and ($\Diamond$7) show that ($\Rightarrow$4) and ($\Rightarrow$7) are equivalent. Assume ($\Rightarrow$7) and $\alpha \preccurlyeq x \Rightarrow y, \beta$. Then applying the exchange law and strong cut law, we have $x\alpha \preccurlyeq y\beta$. Thus ($\Rightarrow$7) implies ($\Rightarrow$5). Assume ($\Rightarrow$7) and $\alpha \preccurlyeq x\beta,\ y\alpha \preccurlyeq \beta$. Then applying the strong cut law, exchange law, and contraction law, we have $x \Rightarrow y, \alpha \preccurlyeq \beta$. Thus ($\Rightarrow$7) implies ($\Rightarrow$6). We have $x \Rightarrow y \preccurlyeq x \Rightarrow y$ by the repetition law, and so ($\Rightarrow$5) with $\alpha = x \Rightarrow y$ and $\beta = \varepsilon$ implies ($\Rightarrow$7). We have $x \preccurlyeq xy$ and $yx \preccurlyeq y$ by the repetition law, weakening law, and exchange law, and so ($\Rightarrow$6) with $\alpha = x$ and $\beta = y$ yields $x \Rightarrow y, x \preccurlyeq y$, hence $x, x \Rightarrow y \preccurlyeq y$ by the exchange law. Thus ($\Rightarrow$6) implies ($\Rightarrow$7). □

Theorem 3.15. Assume that $\preccurlyeq$ satisfies the cut law, the strong negation law with respect to a unary operation $\Diamond$ on A, and either of the strong junction laws with respect to binary operations $\wedge, \vee$ on A. Then $\preccurlyeq$ satisfies the strong cut law.

Proof. Consider the case where $\preccurlyeq$ satisfies the strong conjunction law. Assume $\alpha \preccurlyeq x\gamma$ and $x\beta \preccurlyeq \delta$. Then $\alpha\beta \preccurlyeq x\delta\gamma$ and $x\alpha\beta \preccurlyeq \delta\gamma$ by the weakening law and exchange law, and applying the strong negation law to $x\alpha\beta \preccurlyeq \delta\gamma$, we have $\alpha\beta \preccurlyeq x^\Diamond\delta\gamma$, so $\alpha\beta \preccurlyeq x^\Diamond \wedge x, \delta\gamma$ by the strong conjunction law. Also, $x^\Diamond \wedge x \preccurlyeq \varepsilon$ by the repetition law, strong negation law, and strong conjunction law. Therefore $\alpha\beta \preccurlyeq \delta\gamma$ by the cut law. Thus $\preccurlyeq$ satisfies the strong cut law. □

Theorem 3.16. Assume that $\preccurlyeq$ satisfies the cut law and the strong implication law with respect to a binary operation $\Rightarrow$ on A. Then $\preccurlyeq$ satisfies the strong cut law.

Proof. Assume $\alpha \preccurlyeq x\gamma$ and $x\beta \preccurlyeq \delta$. Then $\alpha\beta \preccurlyeq x\delta\gamma$ and $x\alpha\beta \preccurlyeq \delta\gamma$ by the weakening law and exchange law, and so $x \Rightarrow x, \alpha\beta \preccurlyeq \delta\gamma$ by the strong implication law. Also, $\varepsilon \preccurlyeq x \Rightarrow x$ by the repetition law and strong implication law. Therefore $\alpha\beta \preccurlyeq \delta\gamma$ by the cut law. Thus $\preccurlyeq$ satisfies the strong cut law. □

Theorem 3.17. Let $x \wedge y$, $x \vee y$, $x^{\Diamond}$, $x \Rightarrow y$ be operations on A. Then the following conditions are equivalent.

(1) The $\preccurlyeq$ is Boolean with respect to $\wedge, \vee, \Diamond, \Rightarrow$.
(2) The $\preccurlyeq$ is weakly Boolean with respect to $\wedge, \vee, \Diamond, \Rightarrow$ and satisfies the cut law.

Proof. This is a consequence of Theorems 3.12, 3.13, 3.14, and 3.15 or 3.16. □

Theorem 3.18. Assume that $\preccurlyeq$ satisfies the cut law. Then according as $\preccurlyeq$ satisfies the quasi-conjunction law or the quasi-disjunction law, $\preccurlyeq$ satisfies the following n-tuple quasi-conjunction law or the n-tuple quasi-disjunction law for each $n \geq 1$:

for each element $(x_1, \ldots, x_n) \in A^n$, there exists an element $y \in A$ such that
$y \preccurlyeq x_i$ $(i = 1, \ldots, n)$, and $x_1 \cdots x_n \preccurlyeq y$, ($n$-tuple quasi-conjunction law)
for each element $(x_1, \ldots, x_n) \in A^n$, there exists an element $y \in A$ such that
$y \succcurlyeq x_i$ $(i = 1, \ldots, n)$, and $x_1 \cdots x_n \succcurlyeq y$. ($n$-tuple quasi-disjunction law)

Proof is a routine using induction and the cut law.

3.3. Restrictions of and extensions to latticed relations

The following theorem obviously holds.

Theorem 3.19. Let A be a set and $\preccurlyeq$ be a latticed relation on A^*. Let $\vDash$ be the restriction of $\preccurlyeq$ to $A^* \times A$. Then $\vDash$ is a **partially latticed relation** in the sense that it satisfies the following five laws:

$$x \vDash x, \qquad \text{(repetition law)}$$
$$\alpha \vDash y \implies x\alpha \vDash y, \qquad \text{(partial weakening law)}$$
$$xx\alpha \vDash y \implies x\alpha \vDash y, \qquad \text{(partial contraction law)}$$
$$\alpha xy\beta \vDash z \implies \alpha yx\beta \vDash z, \qquad \text{(partial exchange law)}$$
$$\alpha \vDash x,\ x\beta \vDash y \implies \alpha\beta \vDash y. \qquad \text{(partial cut law)}$$

Furthermore, the restriction of $\preccurlyeq$ to $A \times A$ is a preorder.

Theorem 3.20. Let A be a set and $\vDash$ be a partially latticed relation between A^* and A. Define the relation $\preccurlyeq$ on A^* so that $\alpha \preccurlyeq y_1 \cdots y_n$ holds iff the following holds for all elements $z \in A$:

$$y_i \vDash z \ (i = 1, \ldots, n) \implies \alpha \vDash z.$$

Then $\preccurlyeq$ satisfies the lower quasi-end law and is the largest of the latticed relations on A^* which extend $\vDash$ (therefore we call $\preccurlyeq$ the **largest latticed extension** of $\vDash$).

Proofs of this and the next theorem are similar routines.

Theorem 3.21. Let A be a set and $\sqsubseteq$ be a preorder on A. Define the relation $\preccurlyeq$ on A^* so that $x_1 \cdots x_m \preccurlyeq y_1 \cdots y_n$ holds iff the following holds for all elements $(x, y) \in A \times A$:

$$x \sqsubseteq x_i \ (i = 1, \ldots, m),\ y_j \sqsubseteq y \ (j = 1, \ldots, n) \implies x \sqsubseteq y.$$

Then $\preccurlyeq$ satisfies the quasi-end laws and is the largest of the latticed relations on A^* which extend $\sqsubseteq$ (therefore we call $\preccurlyeq$ the **largest latticed extension** of $\sqsubseteq$).

Remark 3.5. Theorem 3.11 shows that, under a certain condition on a latticed representation $(A, \mathbb{B}, f)$, the f-validity relation $\preccurlyeq_f$ is equal to the largest latticed extension of the preorder $\leq_f$ obtained by restricting $\preccurlyeq_f$ to $A \times A$.

Theorem 3.22. Let A be a set and $\preccurlyeq$ be a latticed relation on A^*. Assume that $\preccurlyeq$ satisfies the quasi-disjunction law and lower quasi-end law. Then $\preccurlyeq$ is equal to the largest latticed extension of the partially latticed relation $\vDash$ obtained by restricting $\preccurlyeq$ to $A^* \times A$.

Proof. Let $\vDash^*$ denote the largest latticed extension of $\vDash$. Since $\preccurlyeq \subseteq \vDash^*$ and both $\preccurlyeq$ and $\vDash^*$ satisfy the lower quasi-end law, we only need to show that if $\alpha \vDash^* y_1 \cdots y_n$ with $n \geq 1$, then $\alpha \preccurlyeq y_1 \cdots y_n$. Since $\preccurlyeq$ satisfies the quasi-disjunction law, Theorem 3.18 shows that there exists an element $z \in A$ such that $y_j \preccurlyeq z$ $(i = 1, \ldots, n)$ and $z \preccurlyeq y_1 \cdots y_n$. Since $\preccurlyeq$ is an extension of $\vDash$, we have $y_j \vDash z$ $(i = 1, \ldots, n)$, and so $\alpha \vDash z$ by the definition of $\vDash^*$, hence $\alpha \preccurlyeq z$. Applying the cut law to $\alpha \preccurlyeq z$ and $z \preccurlyeq y_1 \cdots y_n$, we conclude that $\alpha \preccurlyeq y_1 \cdots y_n$ holds as desired. $\square$

Theorem 3.23. Let A be a set and $\preccurlyeq$ be a latticed relation on A^*. Assume that $\preccurlyeq$ satisfies the quasi-junction laws and quasi-end laws. Then $\preccurlyeq$ is equal to the largest latticed extension of the preorder $\sqsubseteq$ on A obtained by restricting $\preccurlyeq$ to $A \times A$.

Proof is quite similar to that of Theorem 3.22 and left to the reader.

Theorem 3.24. Let A be a set and $\preccurlyeq$ be a latticed relation on A^*. Assume that $\preccurlyeq$ is equal to the largest latticed extension of the partially latticed relation $\vDash$ obtained by restricting $\preccurlyeq$ to $A^* \times A$ and that $\preccurlyeq$ satisfies the following laws with respect to binary operations $x \vee y$ and $x \Rightarrow y$ on A:

$$\left.\begin{array}{l} x \preccurlyeq x \vee y, \\ y \preccurlyeq x \vee y, \end{array}\right\} \tag{3.4}$$

$$x \vee y, x \Rightarrow z, y \Rightarrow z \preccurlyeq z, \tag{3.5}$$

$$x\alpha \preccurlyeq y \implies \alpha \preccurlyeq x \Rightarrow y. \tag{deduction law}$$

Then $\preccurlyeq$ satisfies the strong disjunction law with respect to the operation $\vee$.

Proof. Assume $x\alpha \preccurlyeq z$ and $y\alpha \preccurlyeq z$. Then, we have $\alpha \preccurlyeq x \Rightarrow z$ and $\alpha \preccurlyeq y \Rightarrow z$ by the deduction law. By repeated application of the cut law, exchange law, and contraction law to these two relations and (3.5), we have $x \vee y, \alpha \preccurlyeq z$. Thus $\preccurlyeq$ satisfies the following law:

$$x\alpha \preccurlyeq z,\ y\alpha \preccurlyeq z \implies x \vee y, \alpha \preccurlyeq z. \tag{3.6}$$

This law with $\alpha = \varepsilon$ becomes "$x \vDash z,\ y \vDash z \implies x \vee y \vDash z$," because $\vDash$ is the restriction of $\preccurlyeq$ to $A^* \times A$. Therefore $\preccurlyeq$ satisfies the law $x \vee y \preccurlyeq xy$, because $\preccurlyeq$ is the largest latticed extension of $\vDash$. This law and (3.4) constitute the disjunction law. Therefore, $\preccurlyeq$ satisfies the union

$$\alpha \preccurlyeq xy\beta \iff \alpha \preccurlyeq x \vee y, \beta \tag{3.7}$$

of the laws ($\vee$2) and ($\vee$5) of Theorem 3.12. This implies one of the two laws which constitute the strong disjunction law. It remains to prove another:

$$x\alpha \preccurlyeq \beta,\ y\alpha \preccurlyeq \beta \implies x \vee y, \alpha \preccurlyeq \beta. \tag{3.8}$$

In proving this, we may assume $\beta \neq \varepsilon$, because $\preccurlyeq$ satisfies the lower quasi-end law by Theorem 3.20. Also, (3.7) implies that, when $\beta \neq \varepsilon$, (3.8) is equivalent to (3.6). Therefore $\preccurlyeq$ satisfies (3.8). □

3.4. Latticed representations for latticed relations

Theorem 3.25. Let A be a set and $\preccurlyeq$ be a latticed relation on A^*. Assume that $\preccurlyeq$ satisfies the quasi-junction laws and end laws. Then there exists a latticed representation $(A, \mathbb{B}, f)$ such that the f-validity relation $\preccurlyeq_f$ is equal to $\preccurlyeq$ and $fA = \mathbb{B}$ (note that the converse is also true by Theorems 3.3 and 3.6).

Proof. Let $\sqsubseteq$ be the restriction of $\preccurlyeq$ to $A \times A$. Then $\sqsubseteq$ is a preorder by Theorem 3.19. Let $\equiv$ be the relation on A defined so that $x \equiv y$ iff $x \sqsubseteq y$ and $x \sqsupseteq y$. Then $\equiv$ is an equivalence relation. Let f be the projection of A onto the quotient $\mathbb{B} = A/{\equiv}$. Then $fA = \mathbb{B}$, and we can define the order $\leq$ on $\mathbb{B}$ so that $fx \leq fy$ iff $x \sqsubseteq y$.

Since $\preccurlyeq$ satisfies the end laws, there exist elements $\underline{x}, \overline{x} \in A$ such that $\underline{x} \preccurlyeq \varepsilon$ and $\varepsilon \preccurlyeq \overline{x}$. By the weakening law, every element $x \in A$ satisfies $\underline{x} \preccurlyeq x \preccurlyeq \overline{x}$, hence $f\underline{x} \leq fx \leq f\overline{x}$. Thus $f\underline{x}$ and $f\overline{x}$ are the smallest and the largest elements of $\mathbb{B}$.

Let x, y be arbitrary elements of A. Then by the quasi-conjunction law, there exists an element $z \in A$ such that $z \preccurlyeq x$, $z \preccurlyeq y$, and $xy \preccurlyeq z$, hence $fz \leq fx$ and $fz \leq fy$. Conversely, if an element $z' \in A$ satisfies $fz' \leq fx$ and $fz' \leq fy$, then $z' \preccurlyeq x$ and $z' \preccurlyeq y$, and applying the cut law, exchange law, and contraction law to $z' \preccurlyeq x$, $z' \preccurlyeq y$, and $xy \preccurlyeq z$, we have $z' \preccurlyeq z$, hence $fz' \leq fz$. This implies $fz = \inf\{fx, fy\}$. Similarly by the quasi-disjunction law, there exists an element $w \in A$ such that $w \succcurlyeq x$, $w \succcurlyeq y$, and $xy \succcurlyeq w$, which satisfies $fw = \sup\{fx, fy\}$.

Thus, $\mathbb{B}$ is a lattice which has the smallest element and the largest element, and f is a latticed representation of A on $\mathbb{B}$. Moreover since $fA = \mathbb{B}$, $(A, \mathbb{B}, f)$ satisfies (3.2) for each element $\alpha \in A^*$, and so by Theorem 3.11, $\preccurlyeq_f$ is equal to the largest latticed extension of the restriction $\leq_f$ of $\preccurlyeq_f$ to $A \times A$ (cf. Remark 3.5). On the other hand, since $\preccurlyeq$ satisfies the quasi-end laws by Remark 3.3, Theorem 3.23 shows that $\preccurlyeq$ is equal to the largest latticed extension of $\sqsubseteq$. Since $\sqsubseteq$ is equal to $\leq_f$ by the definition of $\leq$, we conclude that $\preccurlyeq$ is equal to $\preccurlyeq_f$. □

Theorem 3.26. Let A be a set and $\preccurlyeq$ be a latticed relation on A^*. Assume that $\preccurlyeq$ satisfies the end laws. Also, let $x \wedge y$ and $x \vee y$ be binary operations on A, and assume that $\preccurlyeq$ satisfies the strong conjunction law and disjunction law or satisfies the conjunction law and strong disjunction law with respect to the operations $\wedge, \vee$. Then $\preccurlyeq$ satisfies the strong cut law.

Proof. Theorem 3.12 shows that $\preccurlyeq$ satisfies the junction laws with respect to $\wedge, \vee$, and so satisfies the quasi-junction laws. Also, Theorem 3.25 shows that there exists a latticed representation f of A onto a lattice $\mathbb{B}$ such that $\preccurlyeq$ is equal to the f-validity relation $\preccurlyeq_f$. Since $fA = \mathbb{B}$, Theorem 3.10 shows that $\mathbb{B}$ is distributive. Therefore, Theorem 3.2 shows that $\preccurlyeq$ satisfies the strong cut law. $\square$

3.5. Specific latticed relations

Theorem 3.27. Let A be a set and B be a subset of A. Define the relation $\preccurlyeq_B$ on A^* so that $\alpha \preccurlyeq_B \beta$ iff $\alpha \nsubseteq B$ or $\beta \cap B \neq \emptyset$. Then $\preccurlyeq_B$ is a strongly latticed relation.

Proof. Let 1_B be the characteristic mapping of B. Then 1_B is a latticed representation of A on $\mathbb{T} = \{0, 1\}$ and $\preccurlyeq_B$ is equal to the 1_B-validity relation $\preccurlyeq_{1_B}$. Since $\mathbb{T}$ is distributive, $\preccurlyeq_B$ is a strongly latticed relation by Theorems 3.1 and 3.2. $\square$

Theorem 3.28. Let A be a set, φ be a closure operator on $(\mathcal{P}A, \subseteq)$, and define the relation $\preccurlyeq_\varphi$ on A^* so that $\alpha \preccurlyeq_\varphi \beta$ iff $\varphi\alpha \supseteq \bigcap_{y \in \beta} \varphi\{y\}$, where if $\beta = \varepsilon$ then $\bigcap_{y \in \beta} \varphi\{y\} = A$. Then $\preccurlyeq_\varphi$ is a latticed relation.

Proof. Let $\mathcal{B}$ be the fixture domain of φ, define the order $\leq$ on $\mathcal{B}$ so that $X \leq Y$ iff $X \supseteq Y$, and define the mapping $f \in A \to \mathcal{B}$ by $fx = \varphi\{x\}$. Then by Theorem 2.10, $\mathcal{B}$ is a lattice with $\min \mathcal{B} = A$, $\max \mathcal{B} = \varphi\emptyset$, and f is a latticed representation of A on $\mathcal{B}$. Let $\alpha, \beta \in A^*$ and define $\mathcal{X} = \left\{\varphi\{x\} \mid x \in \alpha\right\}$, $\mathcal{Y} = \left\{\{x\} \mid x \in \alpha\right\}$, $\mathcal{Z} = \left\{\varphi\{y\} \mid y \in \beta\right\}$. Then $\mathcal{X}, \mathcal{Z} \subseteq \mathcal{B}$, $\mathcal{Y} \subseteq \mathcal{P}A$, and the following holds:

$$
\begin{aligned}
f\alpha &= \{fx \mid x \in \alpha\} = \left\{\varphi\{x\} \mid x \in \alpha\right\} = \mathcal{X} = \{\varphi\eta \mid \eta \in \mathcal{Y}\} = \varphi\mathcal{Y}, \\
f\beta &= \{fy \mid y \in \beta\} = \left\{\varphi\{y\} \mid y \in \beta\right\} = \mathcal{Z}.
\end{aligned}
$$

Therefore the following holds by Theorem 2.10:

$$
\begin{aligned}
&\inf_{\mathcal{B}} f\alpha = \varphi\left(\sup_{\mathcal{P}A} \mathcal{X}\right) = \varphi\left(\sup_{\mathcal{P}A} \varphi\mathcal{Y}\right) = \varphi\left(\sup_{\mathcal{P}A} \mathcal{Y}\right) = \varphi\alpha, \\
&\sup_{\mathcal{B}} f\beta = \inf_{\mathcal{P}A} \mathcal{Z} = \textstyle\bigcap_{y \in \beta} \varphi\{y\}.
\end{aligned}
$$

Therefore, $\preccurlyeq_\varphi$ is equal to the f-validity relation $\preccurlyeq_f$ and so is a latticed relation by Theorems 3.1 and 3.2. $\square$

Theorem 3.29. Let A be a set and B be a subset of A. Define the relation $\vDash_B$ between A^* and A so that $\alpha \vDash_B y$ iff $\alpha \nsubseteq B$ or $y \in B$. Then $\vDash_B$ is a partially latticed relation.

Proof. This is a consequence of Theorems 3.27 and 3.19, because $\vDash_B$ is the restriction of the relation $\preccurlyeq_B$ defined in Theorem 3.27. $\square$

Theorem 3.30. Let A be a set, φ be a closure operator on $(\mathcal{P}A, \subseteq)$, and define the relation $\vDash_\varphi$ between A^* and A so that $\alpha \vDash_\varphi y$ iff $\varphi\alpha \ni y$. Then $\vDash_\varphi$ is a partially latticed relation.

Proof. Since φ is a closure operator, it follows that $\varphi\alpha \ni y$ iff $\varphi\alpha \supseteq \varphi\{y\}$. Therefore, $\vDash_\varphi$ is the restriction of the relation $\preccurlyeq_\varphi$ defined in Theorem 3.28, and the result follows from Theorems 3.28 and 3.19. □

Theorem 3.31. Let A be a set and φ be a closure operator on $(\mathcal{P}A, \subseteq)$. Then the relation $\preccurlyeq_\varphi$ defined in Theorem 3.28 is the largest latticed extension of the relation $\vDash_\varphi$ defined in Theorem 3.30.

Proof. This is because an element $(\alpha, y_1 \cdots y_n) \in A^* \times A^*$ satisfies $\varphi\alpha \supseteq \varphi\{y_1\} \cap \cdots \cap \varphi\{y_n\}$ iff $\varphi\{y_i\} \ni z$ $(i = 1, \ldots, n)$ implies $\varphi\alpha \ni z$. □

3.6. Closure of the laws on relations

Definition 3.4. Let S and T be sets, and let L be a law on the relations between S and T. Then L is said to be **$\cap$-closed** if the set of the relations between S and T which satisfy L is $\cap$-closed, when regarded as a subset of $\mathcal{P}(S \times T)$.

Theorem 3.32. All the laws which define the latticed relations, partially latticed relations, Boolean relations, and weakly Boolean relations are $\cap$-closed.

Proof is left to the reader (cf. Example 5.2).

4. Theories for logics

Being ready, here we begin a systematic study of logical spaces and deduction systems. The purpose here is stated right after Definition 4.1.

Let A be a set. Then a **logic** or a **relational logic** on A is a relation R between A^* and A. For the logic R, if a subset B of A satisfies the condition

$$\alpha \subseteq B,\ \alpha R y \implies y \in B \tag{4.1}$$

written by sequence convention, then we call B an **R-theory** or say that B is **closed** under R or that R **closes** B. Obviously, A itself is an R-theory. (4.1) implies

$$\varepsilon R y \implies y \in B.$$

Therefore, defining the **R-core** A_R in A by

$$A_R = \{y \in A \mid \varepsilon R y\},$$

we have that every R-theory contains the R-core.

Remark 4.1. The inference rules $\wp$ and & mentioned in the introduction are logics on the set A of the formulas of PL or FPL. The f-validity relation (3.1) defined by a latticed representation $(A, \mathbb{B}, f)$ yields a logic on A when restricted to $A^* \times A$. Thus, a logic R on a set A is not only a model of an inference rule but also a model of a relation between A^* and A which can hardly be regarded as an inference rule. *This is a key to the theory of logical spaces* and also is the reason why we use the term "logic" instead of "rule" or something like that. Nevertheless, let us regard R as an inference rule for the time being. Then if elements $x_1, \dots, x_n, y$ of A satisfy $x_1 \cdots x_n \, R \, y$, we may think y to **directly follow** from or to be a **direct consequence** of $x_1, \dots, x_n$ by virtue of R (cf. Remark 4.3). Also, a subset B of A is an R-theory iff every direct consequence of elements of B by virtue of R belongs to B.

Theorem 4.1. Let R be a logic on a set A. Then the set of all R-theories in A is $\cap$-closed in $\mathcal{P}A$.

Proof. Let $(B_i)_{i \in I}$ be a family of R-theories and define $B = \bigcap_{i \in I} B_i$. If $\alpha \subseteq B$ and $\alpha \, R \, y$, then for all $i \in I$, since $\alpha \subseteq B_i$ and B_i is an R-theory, we have $y \in B_i$, hence $y \in B$. This argument works even if $I = \emptyset$. Therefore, B is an R-theory. $\square$

Definition 4.1. Let X be a subset of a set A and R be a logic on A. Then Theorem 4.1 shows that there exists the smallest of the R-theories in A which contain X. We denote it by $[X]_R$ and call it the **R-closure** of X (see Remark 4.3 for its meaning).

The purpose of this section is to study logics R and R-theories in terms of the mapping $X \mapsto [X]_R$. The main result is Theorem 4.5. Its rather easy consequence Theorem 4.6 is, however, crucial to the main result Theorem 5.6 of §5.1. Its consequence Theorem 6.3 plays an important role in §8.

Theorem 4.2. Let R be a logic on a set A and X be a subset of A. Then $[X]_R$ is the union $\bigcup_{n \geq 0} X_n$ of the **R-descendants** X_n $(n = 0, 1, \dots)$ of X, where $X_0 = X$ and X_n $(n \geq 1)$ is the inductively defined set of the elements $y \in A$ such that $x_1 \cdots x_m \, R \, y$ for some elements $x_i \in X_{l_i}$ $(i = 1, \dots, m)$ with $n = 1 + \sum_{i=1}^{m} l_i$.

Remark 4.2. If $m = 0$ in the above definition of X_n $(n \geq 1)$, then $x_1 \cdots x_m \, R \, y$ means $\varepsilon \, R \, y$ and $n = 1 + \sum_{i=1}^{m} l_i$ means $n = 1$.

Proof. First we will show that every element $y \in X_n$ $(n = 0, 1, \dots)$ belongs to $[X]_R$ by induction on n. This holds when $n = 0$ because $X_0 = X \subseteq [X]_R$. Assume $n \geq 1$. Then $x_1 \cdots x_m \, R \, y$ for some elements $x_i \in X_{l_i}$ $(i = 1, \dots, m)$ with $n = 1 + \sum_{i=1}^{m} l_i$, and $x_1, \dots, x_m \in [X]_R$ by the induction hypothesis. Therefore $y \in [X]_R$. This argument works even if $m = 0$. We have shown that $B = \bigcup_{n \geq 0} X_n$ is contained in $[X]_R$. Next suppose $x_1, \dots, x_m \in B$ and $x_1 \cdots x_m \, R \, y$. Then for $i = 1, \dots, m$, there exists a non-negative integer l_i such that $x_i \in X_{l_i}$, and defining $n = 1 + \sum_{i=1}^{m} l_i$, we have $y \in X_n$. This argument works even if $m = 0$. Thus B is closed under R. Since $X \subseteq B \subseteq [X]_R$, we conclude that $[X]_R = B$ holds. $\square$

Remark 4.3. In addition to Theorem 4.2, an element $y \in A$ belongs to $[X]_R$ iff there exist elements $x_1, \dots, x_n \in A$ such that $x_n = y$ and, for each $i \in \{1, \dots, n\}$, either $x_i \in X$ or there exist numbers $j_1, \dots, j_k \in \{1, \dots, i-1\}$ satisfying $x_{j_1} \cdots x_{j_k} \, R \, x_i$. Also, $y \in [X]_R$ iff there exists an "R-proof figure" with elements of X at the top and y at the bottom. For this reason, when a subset X and an element y of a set A satisfy $[X]_R \ni y$ for a relation R between A^* and A, we may think y to **indirectly follow** from or to be an **indirect consequence** of X by virtue of R (cf. Remark 4.1).

Theorem 4.3. Let Q, R be logics on a set A, and assume $R \subseteq Q$. Then every Q-theory in A is an R-theory. Consequently, $[X]_R \subseteq [X]_Q$ for every subset X of A.

Proof. Let B be a Q-theory. If $\alpha \subseteq B$ and $\alpha\, R\, y$, then $\alpha\, Q\, y$, and so $y \in B$. Therefore B is an R-theory. Since $[X]_Q$ is an R-theory and $X \subseteq [X]_Q$, Definition 4.1 implies that $[X]_R \subseteq [X]_Q$ holds. □

Theorem 4.4. Let R be a logic on a set A and D be a subset of A. Then the mapping $X \mapsto [X \cup D]_R$ is a finitary closure operator on $(\mathcal{P}A, \subseteq)$, and its fixture domain is equal to the set of the R-theories which contain D. Consequently, the set of the R-theories containing D is $\cap$-closed in $\mathcal{P}A$ and quasi-finitary.

Proof. The latter assertion is derived from the former by Theorems 2.10 (cf. Theorem 4.1) and 2.3. As for the former, we may assume $D = \emptyset$ by virtue of Theorem 2.11. By Definition 4.1, the mapping $X \mapsto [X]_R$ is a closure operator and its fixture domain is equal to the set of the R-theories. Thus, defining $(X)_R = \bigcup_{Y \in \mathcal{P}'X} [Y]_R$, we only need to show $[X]_R \subseteq (X)_R$. We have $X \subseteq (X)_R$, because $\{x\} \subseteq \big[\{x\}\big]_R \subseteq (X)_R$ for each $x \in X$. Suppose $x_1, \ldots, x_n \in (X)_R$ and $x_1 \cdots x_n\, R\, y$. Then there exist sets $Y_1, \ldots, Y_n \in \mathcal{P}'X$ such that $x_i \in [Y_i]_R$ $(i = 1, \ldots, n)$. Define $Y = \bigcup_{i=1}^{n} Y_i$. Then $Y \in \mathcal{P}'X$ and $x_i \in [Y_i]_R \subseteq [Y]_R$ $(i = 1, \ldots, n)$, so $y \in [Y]_R \subseteq (X)_R$. This argument works even if $n = 0$. Thus $(X)_R$ is closed under R, and therefore $[X]_R \subseteq (X)_R$. □

Theorem 4.5 (fundamental theorem of logics and theories). Let A be a set and R be a partially latticed logic on A. Then the following holds for each subset X of A and for each element $\alpha \in A^*$ (see Remarks 4.1 and 4.3 for its meaning):

$$\begin{aligned}
[X]_R &= \{y \in A \mid \text{there exists an element } \alpha \in A^* \text{ such that } \alpha \subseteq X \text{ and } \alpha\, R\, y\} \\
&= \text{the 1st } R\text{-descendant } X_1 \text{ of } X, \\
[\alpha]_R &= \{y \in A \mid \alpha\, R\, y\}. \text{ Consequently } A_R = [\emptyset]_R.
\end{aligned}$$

Proof. As for the first equation, we have $X \subseteq X_1$ by the repetition law and $X_1 \subseteq [X]_R$ by Theorem 4.2, and so we only need to show that X_1 is closed under R. Therefore assume $y_1, \ldots, y_n \in X_1$ and $y_1 \cdots y_n\, R\, z$. Then, for each $i \in \{1, \ldots, n\}$, there exists an element $\alpha_i \in A^*$ such that $\alpha_i \subseteq X$ and $\alpha_i\, R\, y_i$. By repeated application of the partial cut law and partial exchange law to the above $n + 1$ R-relations, we have $\alpha_1 \cdots \alpha_n\, R\, z$, hence $z \in X_1$ as desired. This argument works even if $n = 0$. The second equation is derived from the first by the partial weakening law, partial contraction law, and partial exchange law. □

Theorem 4.6. Let A be a set and Q, R be logics on A. Assume that Q is a partially latticed relation. Then the following three conditions are equivalent.

(1) $R \subseteq Q$.
(2) $[X]_R \subseteq [X]_Q$ for every subset X of A.
(3) Every Q-theory in A is an R-theory.

Proof. Theorem 4.3 says that (1) implies (3). If (3) holds, then $[X]_Q$ is an R-theory containing X for each subset X of A, and so (2) holds. Suppose (2) holds.

If $(\alpha, y) \in A^* \times A$ satisfies $\alpha \, R \, y$, then $y \in [\alpha]_R$, so $y \in [\alpha]_Q$ by (2), hence $\alpha \, Q \, y$ by Theorem 4.5. Therefore (1) holds. □

5. Deduction systems

A **deduction system** on a set A is a pair (R, D) of a logic R on A and a subset D of A. Here we study it abstractly, take a novel view of it, and have its example.

5.1. Deduction relations

Here we study deduction systems in terms of the relations defined below. The main result is Theorem 5.6 on a certain preorder between deduction systems. Its rather easy consequence Theorem 5.7 is, however, crucial to the definition 7.1 of the completeness and related concepts for deduction systems on logical spaces.

For each deduction system (R, D) on a set A, we define the logic R^D on A by

$$\alpha \, R^D \, y \iff [\alpha \cup D]_R \ni y \tag{5.1}$$

for each $(\alpha, y) \in A^* \times A$. We call R^D the D**-closure** of R. We will denote R^D also by $\vDash_{R,D}$ and call it the **partial deduction relation** determined by (R, D) in order to relate R^D to the **deduction relation** $\preccurlyeq_{R,D}$ on A^* defined by

$$\alpha \preccurlyeq_{R,D} \beta \iff [\alpha \cup D]_R \supseteq \textstyle\bigcap_{y \in \beta} \big[\{y\} \cup D\big]_R \tag{5.2}$$

for each $(\alpha, \beta) \in A^* \times A^*$. In fact, $\vDash_{R,D}$ is the restriction of $\preccurlyeq_{R,D}$ to $A^* \times A$.

Remark 5.1. An interpretation of $\alpha \, R^D \, y$ or $\alpha \vDash_{R,D} y$ has been given by Remark 4.3, that is, $\alpha \, R^D \, y$ iff y indirectly follows from $\alpha \cup D$ by virtue of R. However, this kind of interpretation of $\alpha \preccurlyeq_{R,D} \beta$ is needless. What is important about $\preccurlyeq_{R,D}$ is that it is the largest latticed extension of $\vDash_{R,D}$ as will be proved in Theorem 5.2.

Theorem 5.1. Let (R, D) be a deduction system on a set A. Then the following holds for the D-closure R^D of R and the R^D-core A_{R^D}.

(1) $R \subseteq R^D$.
(2) $A_{R^D} = [D]_R$, hence in particular $D \subseteq A_{R^D}$.
(3) If a deduction system (Q, C) on A satisfies $R \subseteq Q$ and $D \subseteq C$, then $R^D \subseteq Q^C$.

Proof. If $(\alpha, y) \in A^* \times A$ satisfies $\alpha \, R \, y$, then $y \in [\alpha]_R$, and since $[\alpha]_R \subseteq [\alpha \cup D]_R$ by Theorem 4.4, we have $\alpha \, R^D \, y$. Thus (1) holds. (2) is a consequence of (5.1) with $\alpha = \varepsilon$. If $(\alpha, y) \in A^* \times A$ satisfies $\alpha \, R^D \, y$, then $y \in [\alpha \cup D]_R$, and since $[\alpha \cup D]_R \subseteq [\alpha \cup C]_Q$ by Theorems 4.3 and 4.4, we have $\alpha \, Q^C \, y$. Thus (3) holds. □

Theorem 5.2. Let (R, D) be a deduction system on a set A. Then R^D is a partially latticed relation, and $\preccurlyeq_{R,D}$ is the largest latticed extension of R^D.

Proof. Define $\varphi \in \mathcal{P}A \to \mathcal{P}A$ by $\varphi X = [X \cup D]_R$. Then φ is a closure operator by Theorem 4.4, and R^D is equal to the relation $\vDash_\varphi$ defined in Theorem 3.30, and $\preccurlyeq_{R,D}$ is equal to the relation $\preccurlyeq_\varphi$ defined in Theorem 3.28. Therefore Theorem 5.2 is a consequence of Theorems 3.30 and 3.31. □

Theorem 5.3. If (R, D) is a deduction system on a set A, then the following holds.

(1) $[\mathsf{X}]_{\mathsf{R}^\mathsf{D}} = [\mathsf{X} \cup \mathsf{D}]_\mathsf{R}$ for every subset X of A.
(2) The set of the R^D-theories in A is equal to that of the R-theories which contain D.

Consequently, if R is a logic on A, then $[\mathsf{X}]_{\mathsf{R}^\emptyset} = [\mathsf{X}]_\mathsf{R}$ for every subset X of A, and the set of the $\mathsf{R}^\emptyset$-theories in A is equal to that of the R-theories.

Proof. Since R^D is a partially latticed relation by Theorem 5.2, Theorem 4.5 implies $[\mathsf{X}]_{\mathsf{R}^\mathsf{D}} = \bigcup_{\mathsf{Y} \in \mathcal{P}'\mathsf{X}} [\mathsf{Y} \cup \mathsf{D}]_\mathsf{R}$. Since the mapping $\mathsf{X} \mapsto [\mathsf{X} \cup \mathsf{D}]_\mathsf{R}$ is finitary by Theorem 4.4, the right-hand side is equal to $[\mathsf{X} \cup \mathsf{D}]_\mathsf{R}$. Thus (1) holds. Also by Theorem 4.4, the fixture domains of the mappings $\mathsf{X} \mapsto [\mathsf{X}]_{\mathsf{R}^\mathsf{D}}$ and $\mathsf{X} \mapsto [\mathsf{X} \cup \mathsf{D}]_\mathsf{R}$ are respectively equal to the set of the R^D-theories and that of the R-theories which contain D. Therefore (2) is a consequence of (1). □

Theorem 5.4. Let (R, D) be a deduction system on a set A. Then in order that $\mathsf{R}^\mathsf{D} = \mathsf{R}$ holds, either of the following conditions is necessary and sufficient.

(1) R is a partially latticed relation and every R-theory in A contains D.
(2) R is a partially latticed relation and $\mathsf{D} \subseteq \mathsf{A}_\mathsf{R}$.

Consequently, a logic R on A satisfies $\mathsf{R}^\emptyset = \mathsf{R}$ iff R is a partially latticed relation.

Proof. Since R^D is a partially latticed relation by Theorem 5.2, $\mathsf{R}^\mathsf{D} = \mathsf{R}$ holds only if R is a partially latticed relation. Therefore assume that R is a partially latticed relation. Then by Theorems 4.6 and 5.3, $\mathsf{R}^\mathsf{D} = \mathsf{R}$ holds iff every R-theory contains D. Since $[\emptyset]_\mathsf{R} = \mathsf{A}_\mathsf{R}$ by Theorem 4.5, every R-theory contains D iff $\mathsf{D} \subseteq \mathsf{A}_\mathsf{R}$. □

Theorem 5.5. Let (R, D) be a deduction system on a set A. Then R^D is the smallest of the logics Q on A which satisfy the following conditions.

(1) Q is a partially latticed relation.
(2) $\mathsf{R} \subseteq \mathsf{Q}$ and $\mathsf{D} \subseteq \mathsf{A}_\mathsf{Q}$.

Proof. If $\mathsf{Q} = \mathsf{R}^\mathsf{D}$, then Theorems 5.1 and 5.2 show that (1) and (2) hold. Conversely if (1) and (2) hold, then $\mathsf{R}^\mathsf{D} \subseteq \mathsf{Q}^\mathsf{D} = \mathsf{Q}$ by Theorems 5.1 and 5.4. □

Theorem 5.6 (fundamental theorem of deduction systems). Let (Q, C), (R, D) be deduction systems on a set A. Then the following four conditions are equivalent.

(1) $\mathsf{R}^\mathsf{D} \subseteq \mathsf{Q}^\mathsf{C}$.
(2) $[\mathsf{X} \cup \mathsf{D}]_\mathsf{R} \subseteq [\mathsf{X} \cup \mathsf{C}]_\mathsf{Q}$ for every subset X of A.
(3) Every Q-theory in A containing C is an R-theory containing D.
(4) $\mathsf{R} \subseteq \mathsf{Q}^\mathsf{C}$ and $\mathsf{D} \subseteq \mathsf{A}_{\mathsf{Q}^\mathsf{C}}$.

Proof. Since Q^C is a partially latticed relation by Theorem 5.2, Theorems 4.6 and 5.3 show that (1)–(3) are equivalent, and Theorem 5.5 shows that (4) implies (1). Conversely under (1), we have $\mathsf{R} \subseteq \mathsf{R}^\mathsf{D} \subseteq \mathsf{Q}^\mathsf{C}$ and $\mathsf{D} \subseteq \mathsf{A}_{\mathsf{R}^\mathsf{D}} \subseteq \mathsf{A}_{\mathsf{Q}^\mathsf{C}}$ by Theorem 5.1, and thus (4) holds. □

Theorem 5.7. Let (R, D) be a deduction system on A and Q be a partially latticed logic on A. Then the following four conditions are equivalent.

(1) $R^D \subseteq Q$.
(2) $[X \cup D]_R \subseteq [X]_Q$ for every subset X of A.
(3) Every Q-theory in A is an R-theory containing D.
(4) $R \subseteq Q$ and $D \subseteq A_Q$.

Also, the following three conditions are equivalent.

(5) $Q \subseteq R^D$.
(6) $[X]_Q \subseteq [X \cup D]_R$ for every subset X of A.
(7) Every R-theory in A containing D is a Q-theory.

Therefore, the following four conditions are equivalent.

(8) $Q = R^D$.
(9) $[X]_Q = [X \cup D]_R$ for every subset X of A.
(10) The set of the Q-theories in A is equal to the set of the R-theories in A containing D.
(11) $R \subseteq Q,\ D \subseteq A_Q$, and $Q \subseteq R^D$.

Proof. We have $Q^\emptyset = Q$ by Theorem 5.4. Therefore, (1)–(4) are equivalent to (1)–(4) of Theorem 5.6 with $C = \emptyset$. Also, (5)–(7) are equivalent to (1)–(3) of Theorem 5.6 with (Q, C) and (R, D) interchanged first and C replaced by $\emptyset$ next. $\square$

5.2. Generational laws for relations

The purpose of this subsection is to view deduction systems on direct products $A \times B$ as laws on relations between A and B. This view is crucial to the fundamental theorem 7.12 of completeness.

Let A, B be sets and (R', D') be a deduction system on the direct product $A' = A \times B$. Then (R', D') is also called a **generational law** on the relations between A and B, and if a relation R between A and B regarded as a subset of A' is closed under R' and contains D', we say that R **satisfies** (R', D') or call R an (R', D')**-relation**. This definition is justified shortly by the examples below.

As an immediate consequence of the above definition, we have that the R'-closure $[D']_{R'}$ of D' regarded as a relation between A and B is the smallest (R', D')-relation. Also, Theorem 4.4 shows that generational laws for relations are $\cap$-closed in the sense of Definition 3.4.

Example 5.1. The equivalence law may be regarded as a generational law. Let A be a set and define $A' = A \times A$. Define the subset D' of A' by $D' = \{(x, x) \mid x \in A\}$. Let R' be the union of the logics $S' = \dfrac{(x, y)}{(y, x)}$ and $T' = \dfrac{(x, y)\ (y, z)}{(x, z)}$ on A' each defined by the fractional expression. Let R be a relation on A and regard it as a subset of A'. Then R is reflexive iff R contains D', R is symmetric iff R is closed under S', and R is transitive iff R is closed under T'. Therefore, R is an equivalence relation iff R contains D' and is closed under R'. Thus, a relation on A is an equivalence relation iff it satisfies the generational law (R', D').

Example 5.2. The Boolean law is regarded as a generational law, and so also are the laws listed in Theorem 3.32. Let A be a set and define $\vec{A} = A^* \times A^*$. Denote the elements $(\alpha, \beta) \in \vec{A}$ by $\alpha \to \beta$ or $\beta \leftarrow \alpha$, and call them **sequents**. Let $\vec{R}$ be the union of the following seven logics on $\vec{A}$ each defined by the fractional expression:

$$\frac{\alpha \to \beta}{x\alpha \to \beta}, \qquad \frac{\alpha \leftarrow \beta}{x\alpha \leftarrow \beta}, \qquad \text{(weakening)}$$

$$\frac{xx\alpha \to \beta}{x\alpha \to \beta}, \qquad \frac{xx\alpha \leftarrow \beta}{x\alpha \leftarrow \beta}, \qquad \text{(contraction)}$$

$$\frac{\alpha xy\beta \to \gamma}{\alpha yx\beta \to \gamma}, \qquad \frac{\alpha xy\beta \leftarrow \gamma}{\alpha yx\beta \leftarrow \gamma}, \qquad \text{(exchange)}$$

$$\frac{\alpha \to x\gamma \quad x\beta \to \delta}{\alpha\beta \to \delta\gamma}. \qquad \text{(strong cut)}$$

Let $\vec{D}$ be the set of the sequents of the following form, where $\wedge, \vee, \diamondsuit, \Rightarrow$ are operations on A:

$$x \to x, \qquad \text{(repetition)}$$

$$x \wedge y \to x, \qquad x \wedge y \to y, \qquad xy \to x \wedge y, \qquad \text{(conjunction)}$$

$$x \vee y \leftarrow x, \qquad x \vee y \leftarrow y, \qquad xy \leftarrow x \vee y, \qquad \text{(disjunction)}$$

$$xx^{\diamondsuit} \to \varepsilon, \qquad xx^{\diamondsuit} \leftarrow \varepsilon, \qquad \text{(negation)}$$

$$x^{\diamondsuit} \to x \Rightarrow y, \qquad y \to x \Rightarrow y, \qquad x \Rightarrow y \to x^{\diamondsuit} y. \qquad \text{(implication)}$$

Let R be a relation on A^* and regard R as a subset of $\vec{A}$. Then R contains $\vec{D}$ and is closed under $\vec{R}$ iff R satisfies the repetition law, weakening law, contraction law, exchange law, strong cut law, junction laws, negation laws, and implication laws with respect to the operations $\wedge, \vee, \diamondsuit, \Rightarrow$. Therefore, R satisfies the generational law $(\vec{R}, \vec{D})$ iff R is a Boolean relation with respect to $\wedge, \vee, \diamondsuit, \Rightarrow$.

5.3. Boolean deduction systems

Throughout this subsection, we let A be a non-empty set with operations $x \wedge y$, $x \vee y$, $x^{\diamondsuit}$, $x \Rightarrow y$ on it and (R, D) be a deduction system on A. As in the introduction, we define logics $\wp$ and $\&$ on A by

$$\wp = \frac{x \quad x \Rightarrow y}{y}, \qquad \& = \frac{x \quad y}{x \wedge y}. \tag{5.3}$$

Motivated by the program to be given in Remark 7.3 and Theorem 7.10 to be proved, here we seek conditions on (R, D) under which the deduction relation $\preccurlyeq_{R,D}$ is Boolean or weakly Boolean with respect to the operations $\wedge, \vee, \diamondsuit, \Rightarrow$. Our goal is Corollary 5.9.1 (cf. Example 7.1), although it is one of examples.

Since $\preccurlyeq_{R,D}$ is a latticed relation by Theorem 5.2, the results in §3.2 apply to $\preccurlyeq_{R,D}$. In particular, Theorem 3.17 shows that $\preccurlyeq_{R,D}$ is Boolean iff it is weakly Boolean. Therefore, we only need to seek conditions on (R, D) under which $\preccurlyeq_{R,D}$ satisfies the strong cut law, junction laws, negation laws, and implication laws. Note that if $\alpha \, R \, y$ then $\alpha \preccurlyeq_{R,D} y$ by Theorem 5.1.

Lemma 5.1. If R contains $\wp$, then $\preccurlyeq_{R,D}$ satisfies the following laws:

$$x, x \Rightarrow y \preccurlyeq_{R,D} y, \qquad \text{(cut-implication law)}$$

$$\alpha \preccurlyeq_{R,D} x \Rightarrow y \implies x\alpha \preccurlyeq_{R,D} y. \qquad \text{(reverse deduction law)}$$

Proof. Since $x, x \Rightarrow y \wp y$ and $\wp \subseteq R$, we have $x, x \Rightarrow y \, R \, y$, hence the cut-implication law. The reverse deduction law is derived from the cut-implication law by the cut law and exchange law. □

Theorem 5.8 (deduction theorem). The deduction relation $\preccurlyeq_{R,D}$ satisfies

$$x\alpha \preccurlyeq_{R,D} y \implies \alpha \preccurlyeq_{R,D} x \Rightarrow y, \qquad \text{(deduction law)}$$

provided $\preccurlyeq_{R,D}$ satisfies the laws (1) $\varepsilon \preccurlyeq_{R,D} x \Rightarrow x$, (2) $y \preccurlyeq_{R,D} x \Rightarrow y$ (2nd implication law), and (3) if $z_1 \cdots z_k \, R \, y$ $(k \geq 1)$ then $x \Rightarrow z_1, \cdots, x \Rightarrow z_k \preccurlyeq_{R,D} x \Rightarrow y$. The converse is true provided that R contains $\wp$.

Proof. Assume $x\alpha \preccurlyeq_{R,D} y$. Define $X = \{x\} \cup \alpha \cup D$. Then $[X]_R \ni y$, and so by Theorem 4.2, y belongs to the n-th R-descendant X_n of X for some non-negative integer n. We will show $\alpha \preccurlyeq_{R,D} x \Rightarrow y$ by induction on n. Assume $n = 0$ or $y \in X$. If $y = x$, then the law (1) and the weakening law imply $\alpha \preccurlyeq_{R,D} x \Rightarrow y$. If $y \in \alpha \cup D$, then $\alpha \preccurlyeq_{R,D} y$, and so the law (2) and the cut law imply $\alpha \preccurlyeq_{R,D} x \Rightarrow y$. Therefore assume $n \geq 1$. Then $z_1 \cdots z_k \, R \, y$ for some elements $z_i \in X_{l_i}$ $(i = 1, \dots, k)$ with $n = 1 + \sum_{i=1}^{k} l_i$. If $k = 0$, then $\alpha \, R \, y$ by the weakening law, and so $\alpha \preccurlyeq_{R,D} y$, hence $\alpha \preccurlyeq_{R,D} x \Rightarrow y$ as above. Therefore assume $k \geq 1$. Then $x \Rightarrow z_1, \cdots, x \Rightarrow z_k \preccurlyeq_{R,D} x \Rightarrow y$ by the law (3), and $\alpha \preccurlyeq_{R,D} x \Rightarrow z_i$ $(i = 1, \dots, k)$ by the induction hypothesis. By repeated application of the cut law, exchange law, and contraction law to the above $k+1$ $\preccurlyeq_{R,D}$-relations, we have $\alpha \preccurlyeq_{R,D} x \Rightarrow y$.

Suppose R contains $\wp$ and $\preccurlyeq_{R,D}$ satisfies the deduction law. Then (1) and (2) hold by the repetition law, weakening law, and deduction law. If $z_1 \cdots z_k \, R \, y$ $(k \geq 1)$, then $z_1 \cdots z_k \preccurlyeq_{R,D} y$, and $x, x \Rightarrow z_1, \cdots, x \Rightarrow z_k \preccurlyeq_{R,D} z_i$ $(i = 1, \dots, k)$ by the cut-implication law of Lemmas 5.1 together with the weakening law and exchange law, hence $x \Rightarrow z_1, \cdots, x \Rightarrow z_k \preccurlyeq_{R,D} x \Rightarrow y$ by the cut law, exchange law, contraction law, and deduction law. □

Corollary 5.8.1. The deduction relation $\preccurlyeq_{\wp,D}$ satisfies the deduction law iff $[D]_\wp$ contains all of the elements of A in the following form: (1) $x \Rightarrow x$, (2) $y \Rightarrow (x \Rightarrow y)$, and (3) $\big(z \Rightarrow (x \Rightarrow y)\big) \Rightarrow \big((z \Rightarrow x) \Rightarrow (z \Rightarrow y)\big)$.

Proof. Note that $\varepsilon \preccurlyeq_{\wp,D} x$ iff $x \in [D]_\wp$. Therefore, if $\preccurlyeq_{\wp,D}$ satisfies the deduction law, then the above elements belong to $[D]_\wp$ by Theorem 5.8. Conversely if they belong to $[D]_\wp$, then by the reverse deduction law of Lemmas 5.1, $\preccurlyeq_{\wp,D}$ satisfies the conditions (1)–(3) of Theorem 5.8, and so satisfies the deduction law. □

Lemma 5.2. If $\preccurlyeq_{R,D}$ satisfies the laws (1) $\varepsilon \preccurlyeq_{R,D} x^\Diamond \vee x$, (2) $y \preccurlyeq_{R,D} x \vee y$, and (3) $x^\Diamond \vee y \preccurlyeq_{R,D} x \Rightarrow y$, then $\preccurlyeq_{R,D}$ satisfies the laws (1) and (2) of Theorem 5.8.

Proof. Apply the cut law to (1) and (3) with $x = y$ to get $\varepsilon \preccurlyeq_{R,D} x \Rightarrow x$. Apply the cut law to (2) with x replaced by $x^\Diamond$ and (3) to get $y \preccurlyeq_{R,D} x \Rightarrow y$. □

Lemma 5.3. If R contains &, then $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the laws (1) $xy \preccurlyeq_{\mathsf{R},\mathsf{D}} x \wedge y$ and (2) $x \wedge y, \beta \preccurlyeq_{\mathsf{R},\mathsf{D}} \alpha \implies xy\beta \preccurlyeq_{\mathsf{R},\mathsf{D}} \alpha$.

Proof. Since $xy \,\&\, x \wedge y$ and $\& \subseteq \mathsf{R}$, we have $xy \,\mathsf{R}\, x \wedge y$, hence (1). Apply the cut law to (1) and the premise of (2) to get the conclusion of (2). □

Theorem 5.9. Assume that R contains $\wp \cup \&$. Then $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is a Boolean relation with respect to the operations $\wedge, \vee, \Diamond, \Rightarrow$, provided that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the following eight laws: (1) $\varepsilon \preccurlyeq_{\mathsf{R},\mathsf{D}} x^{\Diamond} \vee x$, (2) $x \wedge y \preccurlyeq_{\mathsf{R},\mathsf{D}} x$, (3) $x \wedge y \preccurlyeq_{\mathsf{R},\mathsf{D}} y$, (4) $x \preccurlyeq_{\mathsf{R},\mathsf{D}} x \vee y$, (5) $y \preccurlyeq_{\mathsf{R},\mathsf{D}} x \vee y$, (6) $x^{\Diamond} \vee y \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rightarrow y$, (7) $x \vee y, x \Rightarrow z, y \Rightarrow z \preccurlyeq_{\mathsf{R},\mathsf{D}} z$, and (8) if $z_1 \cdots z_k \,\mathsf{R}\, y$ $(k \geq 1)$ then $x \Rightarrow z_1, \cdots, x \Rightarrow z_k \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rightarrow y$.

Proof. The $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is a latticed relation by Theorem 5.2 and satisfies the 2nd implication law by Lemma 5.2. Therefore, we need to show that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the strong cut law, junction laws, negation laws, 1st and 3rd implication laws.

We note that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is the largest latticed extension of $\vDash_{\mathsf{R},\mathsf{D}}$ by Theorem 5.2 and so $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the lower quasi-end law by Theorem 3.20. Also, $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the cut-implication law, reverse deduction law, and deduction law by Lemmas 5.1, 5.2 and Theorem 5.8.

The laws (2), (3) and Lemma 5.3 show that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the conjunction law. Also, Theorem 3.24 shows that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the strong disjunction law. Therefore, $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the disjunction law by Theorem 3.12.

We have $x^{\Diamond} \preccurlyeq_{\mathsf{R},\mathsf{D}} x^{\Diamond} \vee y$ and $x^{\Diamond} \vee y \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rightarrow y$ by the laws (4) and (6), and so we have the 1st implication law $x^{\Diamond} \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rightarrow y$ by the cut law.

Applying the reverse deduction law to the 1st implication law, we have $xx^{\Diamond} \preccurlyeq_{\mathsf{R},\mathsf{D}} y$, hence the lower negation law $xx^{\Diamond} \preccurlyeq_{\mathsf{R},\mathsf{D}} \varepsilon$ because $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the lower quasi-end law. Since $\varepsilon \preccurlyeq_{\mathsf{R},\mathsf{D}} x^{\Diamond} \vee x$ by the law (1), we have the upper negation law $\varepsilon \preccurlyeq_{\mathsf{R},\mathsf{D}} xx^{\Diamond}$ by the disjunction law, cut law, and exchange law.

Since $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the junction laws and negation laws, it follows from Theorem 3.12 that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the end laws. As shown above, $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the conjunction law and strong disjunction law. Therefore by Theorem 3.26, $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the strong cut law.

Since $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the strong cut law, negation laws, and cut-implication law, Theorem 3.14 shows that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the 3rd implication law. □

Corollary 5.9.1. Assume $\mathsf{R} = \wp \cup \&$. Then $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is a Boolean relation with respect to the operations $\wedge, \vee, \Diamond, \Rightarrow$, provided that $[\mathsf{D}]_{\mathsf{R}}$ contains all of the elements of A in the following form: (1) $x^{\Diamond} \vee x$, (2) $(x \wedge y) \Rightarrow x$, (3) $(x \wedge y) \Rightarrow y$, (4) $x \Rightarrow (x \vee y)$, (5) $y \Rightarrow (x \vee y)$, (6) $(x^{\Diamond} \vee y) \Rightarrow (x \Rightarrow y)$, (7) $\big((x \Rightarrow z) \wedge (y \Rightarrow z)\big) \Rightarrow \big((x \vee y) \Rightarrow z\big)$, (8) $\big((z \Rightarrow x) \wedge (z \Rightarrow y)\big) \Rightarrow \big(z \Rightarrow (x \wedge y)\big)$, and (9) $\big((z \Rightarrow x) \wedge \big(z \Rightarrow (x \Rightarrow y)\big)\big) \Rightarrow (z \Rightarrow y)$.

Proof. Since $\mathsf{R} = \wp \cup \&$, the elements in the form (8) or (9) are precisely the elements $\big((x \Rightarrow z_1) \wedge \cdots \wedge (x \Rightarrow z_k)\big) \Rightarrow (x \Rightarrow y)$ with $z_1 \cdots z_k \,\mathsf{R}\, y$ $(k \geq 1)$. We may use the reverse deduction law and Lemma 5.3 (2). Therefore, if the elements in the form (1)–(9) belong to $[\mathsf{D}]_{\mathsf{R}}$, we can show that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the laws (1)–(8) of Theorem 5.9, and so $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is a Boolean relation. For instance, since the elements in the form (7) belong to $[D]_R$, we have $\varepsilon \preccurlyeq_{R,D} \big((x \Rightarrow z) \wedge (y \Rightarrow z)\big) \Rightarrow \big((x \vee y) \Rightarrow z\big)$, and it follows that $\preccurlyeq_{R,D}$ satisfies the law (7) of Theorem 5.9. □

6. Logical spaces

A **logical space** is a pair $(A, \mathcal{B})$ of a non-empty set A and a subset $\mathcal{B}$ of $\mathcal{P}A$, which we call the set of the **given theories** of the logical space. A logic R on A is said to be $\mathcal{B}$-**sound** or called a $\mathcal{B}$-**logic**, if every element of $\mathcal{B}$ is closed under R. A subset X of A is called a $\mathcal{B}$-**theory**, if X is closed under every $\mathcal{B}$-logic on A. We call $\bigcap_{B \in \mathcal{B}} B$ the $\mathcal{B}$-**core**. Elements and subsets of A are said to be $\mathcal{B}$-**sound** if they are contained in the $\mathcal{B}$-core. A $\mathcal{B}$-sound element is also called a $\mathcal{B}$-**tautology**.

As immediate consequences of the above definitions, we have that the given theories are $\mathcal{B}$-theories and hence that $\mathcal{B}$-logics are the only logics on A that close every $\mathcal{B}$-theory. A Galois connection underlies these facts.

6.1. Control by the largest logic

The purpose of this subsection is to show that all the above concepts are controlled by the largest $\mathcal{B}$-logic.

Theorem 6.1. Let $(A, \mathcal{B})$ be a logical space. Then the following holds.

(1) There exists the largest $\mathcal{B}$-logic on A,

which we denote by Q for the time being.

(2) A logic R on A is a $\mathcal{B}$-logic iff R is contained in Q.

Let X be a subset of A. Then the following holds.

(3) The X is a $\mathcal{B}$-theory iff X is a Q-theory.
(4) The Q-closure $[X]_Q$ of X is the smallest of the $\mathcal{B}$-theories which contain X.

Proof. Let $\mathcal{R}$ be the set of the $\mathcal{B}$-logics on A, regard $\mathcal{R}$ as a subset of $\mathcal{P}(A^* \times A)$, and define $Q = \bigcup_{R \in \mathcal{R}} R$. If $\alpha \subseteq B \in \mathcal{B}$, $y \in A$, and $\alpha\, Q\, y$, then $\alpha\, R\, y$ for some $R \in \mathcal{R}$, and so $y \in B$. Thus Q is also a $\mathcal{B}$-logic on A and so is the largest one. Suppose $R \subseteq Q$. Then every element $B \in \mathcal{B}$ is a Q-theory and so is an R-theory by Theorem 4.3. Therefore, R is also a $\mathcal{B}$-logic. Since Q is a $\mathcal{B}$-logic, $\mathcal{B}$-theories are Q-theories. Conversely if X is a Q-theory, then X is an R-theory for each $R \in \mathcal{R}$ again by Theorem 4.3, and so is a $\mathcal{B}$-theory. Thus we have proved (1)–(3). Finally, (4) is a direct consequence of (3) and Definition 4.1. □

Theorem 6.2. Let $(A, \mathcal{B})$ be a logical space. Then the following holds for the largest $\mathcal{B}$-logic Q on A and each $(\alpha, y) \in A^* \times A$:

$$\alpha\, Q\, y \iff y \in \textstyle\bigcap_{\alpha \subseteq B \in \mathcal{B}} B.$$

Proof. Define the logic P on A so that $\alpha\, P\, y$ iff $y \in \bigcap_{\alpha \subseteq B \in \mathcal{B}} B$. If $\alpha \subseteq B \in \mathcal{B}$, $y \in A$, and $\alpha\, P\, y$, then $y \in B$ by the definition of P, which shows that P is a $\mathcal{B}$-logic, hence $P \subseteq Q$. If $\alpha\, Q\, y$ and $\alpha \subseteq B \in \mathcal{B}$, then since Q is a $\mathcal{B}$-logic, we have $y \in B$. This shows $Q \subseteq P$. Therefore $Q = P$. □

Theorem 6.3. Let $(A, \mathcal{B})$ be a logical space and Q be the largest $\mathcal{B}$-logic on A. Then Q is a partially latticed relation, and the following holds for each subset X of A and for each element $\alpha \in A^*$:

$$[X]_Q = \{y \in A \mid \text{there exists an element } \alpha \in A^* \text{ such that } \alpha \subseteq X \text{ and } \alpha\, Q\, y\},$$
$$[\alpha]_Q = \{y \in A \mid \alpha\, Q\, y\}.$$

Proof. For each set $B \in \mathcal{B}$, define the logic $\vDash_B$ on A as in Theorem 3.29. Then $Q = \bigcap_{B \in \mathcal{B}} \vDash_B$ by Theorem 6.2. Therefore the former assertion is a consequence of Theorems 3.29 and 3.32. The latter assertion is a consequence of the former and Theorem 4.5. □

Alternative proof. It follows from Theorems 5.3 and 5.1 that $Q^{\emptyset}$ is a $\mathcal{B}$-logic and $Q \subseteq Q^{\emptyset}$. Therefore $Q = Q^{\emptyset}$, and so Q is a partially latticed relation by Theorem 5.2, and the second equation holds. Since the mapping $X \mapsto [X]_Q$ is finitary by Theorem 4.4, the first equation follows from the second. □

Theorem 6.4. Let $(A, \mathcal{B})$ be a logical space and Q be the largest $\mathcal{B}$-logic on A. Then the $\mathcal{B}$-core C of A satisfies $C = A_Q = [\emptyset]_Q$.

Proof. This is a consequence of Theorems 6.2 and 6.3 (cf. Theorem 4.5). □

6.2. Functional logical spaces and their validity relations

A $\mathbb{B}$-valued **functional logical space** is a pair $(A, \mathcal{F})$ of a non-empty set A and a subset $\mathcal{F}$ of $A \to \mathbb{B}$, where $\mathbb{B}$ is a non-trivial lattice which has the smallest element 0 and the largest element 1. For each $f \in \mathcal{F}$ and $a \in \mathbb{B}$, we define

$$A_{f,a} = \{x \in A \mid fx \geq a\}.$$

Notice that $A_{f,0} = A$ and $A_{f,1} = \{x \in A \mid fx = 1\}$. Furthermore, we define

$$\mathcal{B}_{\mathcal{F}} = \begin{cases} \{A_{f,a} \mid f \in \mathcal{F},\ a \in \mathbb{B}\} & \text{when } \mathcal{F} \neq \emptyset, \\ \{A\} & \text{when } \mathcal{F} = \emptyset. \end{cases}$$

Then $(A, \mathcal{B}_{\mathcal{F}})$ is a logical space, and we have defined various concepts for it, that is, the $\mathcal{B}_{\mathcal{F}}$-logics, $\mathcal{B}_{\mathcal{F}}$-theories, $\mathcal{B}_{\mathcal{F}}$-core, and so on. Call them the **$\mathcal{F}$-logics, $\mathcal{F}$-theories, $\mathcal{F}$-core**, and so on, and similarly for the concepts to be introduced.

The purpose of this subsection is to connect the largest $\mathcal{F}$-logic with the values of the functions in $\mathcal{F}$. Theorems 6.5 and 6.6 will help the reader understand the meanings of the basic concepts particularly when $\mathbb{B} = \mathbb{T}$.

We define, for each $f \in \mathcal{F}$, the relation $\vDash_f$ between A^* and A by

$$\alpha \vDash_f y \iff \inf f\alpha \leq fy.$$

We call $\vDash_f$ the **partial f-validity relation**, because it is equal to the restriction to $A^* \times A$ of the f-validity relation $\preccurlyeq_f$ defined by (3.1):

$$\alpha \preccurlyeq_f \beta \iff \inf f\alpha \leq \sup f\beta.$$

Furthermore, we define the relation $\vDash_{\mathcal{F}}$ between A^* and A by

$$\alpha \vDash_{\mathcal{F}} y \iff \alpha \vDash_f y \text{ for every } f \in \mathcal{F},$$

which we call the **partial $\mathcal{F}$-validity relation**. It is equal to the restriction to $A^* \times A$ of the **$\mathcal{F}$-validity relation** $\preccurlyeq_{\mathcal{F}}$ defined by

$$\alpha \preccurlyeq_{\mathcal{F}} \beta \iff \alpha \preccurlyeq_{f} \beta \text{ for every } f \in \mathcal{F}. \tag{6.1}$$

Remark 6.1. The importance of $\vDash_{\mathcal{F}}$ is clarified by the following theorem, while that of $\preccurlyeq_{\mathcal{F}}$ is rather a technical one (cf. Theorem 6.8 and Remark 5.1).

Theorem 6.5. Let $(A, \mathcal{F})$ be a $\mathbb{B}$-valued functional logical space. Then the largest $\mathcal{F}$-logic on A is equal to $\vDash_{\mathcal{F}}$. Also, elements $x_1, \ldots, x_n, y \in A$ satisfy $x_1 \cdots x_n \vDash_{\mathcal{F}} y$ iff they satisfy the following condition for all $(f, a) \in \mathcal{F} \times \mathbb{B}$:

$$fx_1 \geq a, \ \ldots, \ fx_n \geq a \implies fy \geq a. \tag{6.2}$$

Proof. Let Q be the largest $\mathcal{F}$-logic. Then, Q is the largest $\mathcal{B}_{\mathcal{F}}$-logic, and so by Theorem 6.2, elements $x_1, \ldots, x_n, y \in A$ satisfy $x_1 \cdots x_n \, Q \, y$ iff they satisfy (6.2) for all $(f, a) \in \mathcal{F} \times \mathbb{B}$. Furthermore, elements $x_1, \ldots, x_n, y$ and $f \in \mathcal{F}$ satisfy (6.2) for all $a \in \mathbb{B}$ iff they satisfy $\inf\{fx_1, \ldots, fx_n\} \leq fy$, that is, $x_1 \cdots x_n \vDash_f y$. Thus, $x_1, \ldots, x_n, y$ satisfy $x_1 \cdots x_n \, Q \, y$ iff they satisfy $x_1 \cdots x_n \vDash_{\mathcal{F}} y$. $\square$

Theorem 6.6. The following holds for each $\mathbb{B}$-valued functional logical space $(A, \mathcal{F})$.

(1) A logic R on A is an $\mathcal{F}$-logic iff it is contained in $\vDash_{\mathcal{F}}$, that is, iff it satisfies the following condition for all $(f, a) \in \mathcal{F} \times \mathbb{B}$:

$$fx_1 \geq a, \ \ldots, \ fx_n \geq a, \ x_1 \cdots x_n \, R \, y \implies fy \geq a.$$

(2) A subset X of A is an $\mathcal{F}$-theory iff it is closed under $\vDash_{\mathcal{F}}$, that is, iff it satisfies

$$\begin{aligned} &x_1, \ldots, x_n \in X, \ y \in A - X \\ &\implies fx_1 \geq a, \ \ldots, \ fx_n \geq a \text{ but } fy \not\geq a \text{ for some } (f, a) \in \mathcal{F} \times \mathbb{B}. \end{aligned}$$

(3) An element $x \in A$ is an $\mathcal{F}$-tautology iff $\varepsilon \vDash_{\mathcal{F}} x$, that is, iff $fx = 1$ for all $f \in \mathcal{F}$.

Proof. This is a restatement of part of Theorems 6.1 and 6.4 by Theorem 6.5. $\square$

Theorem 6.7. Let $(A, \mathcal{F})$ be a functional logical space. Then $\vDash_{\mathcal{F}}$ is a partially latticed relation, and $\preccurlyeq_{\mathcal{F}}$ is a latticed relation. If $\preccurlyeq_{\mathcal{F}}$ satisfies the quasi-disjunction law and lower quasi-end law, then $\preccurlyeq_{\mathcal{F}}$ is the largest latticed extension of $\vDash_{\mathcal{F}}$.

Proof. Since $\preccurlyeq_f$ is a latticed relation for each $f \in \mathcal{F}$ by Theorems 3.1 and 3.2, so is $\preccurlyeq_{\mathcal{F}}$ by Theorem 3.32. Therefore by Theorem 3.19, $\vDash_{\mathcal{F}}$ is a partially latticed relation. The latter assertion is a consequence of the former and Theorem 3.22. $\square$

Definition 6.1. A **Boolean logical space** is a $\mathbb{B}$-valued functional logical space $(A, \mathcal{F})$ which satisfies the following two conditions:

(1) $\mathbb{B}$ is a Boolean lattice with respect to the meet $\wedge$, join $\vee$, complement $\Diamond$, and implication $\Rightarrow$.
(2) With respect to some operations $x \wedge y$, $x \vee y$, $x^{\Diamond}$, $x \Rightarrow y$ on A, every element of $\mathcal{F}$ is a Boolean representation of A on $\mathbb{B}$.

When $\mathbb{B} = \mathbb{T} = \{0, 1\}$, we call $(A, \mathcal{F})$ a **binary logical space**.

Remark 6.2. Every Boolean logical space $(\mathsf{A}, \mathcal{F})$ is "equivalent" to the binary logical space $(\mathsf{A}, \mathcal{F}')$ made of the set $\mathcal{F}'$ of the composites of the elements of $\mathcal{F}$ and the binary representations of $\mathbb{B}$ (cf. Remark 6.3).

Theorem 6.8. Let $(\mathsf{A}, \mathcal{F})$ be a Boolean logical space with respect to operations $\wedge, \vee, \Diamond, \Rightarrow$ on A. Then $\preccurlyeq_{\mathcal{F}}$ is a Boolean relation with respect to the operations and is the largest latticed extension of $\vDash_{\mathcal{F}}$.

Proof. Since $\preccurlyeq_{\mathsf{f}}$ is Boolean for each $\mathsf{f} \in \mathcal{F}$ by Theorem 3.9, so is $\preccurlyeq_{\mathcal{F}}$ by Theorem 3.32. Therefore, $\preccurlyeq_{\mathcal{F}}$ satisfies the quasi-disjunction law. It also satisfies the lower negation law and conjunction law, hence the lower end law by Theorem 3.12, and hence the lower quasi-end law by Remark 3.3. Thus, $\preccurlyeq_{\mathcal{F}}$ is the largest latticed extension of $\vDash_{\mathcal{F}}$ by Theorem 6.7. □

Example 6.1. Logical spaces in PL, FPL, MPCL, and PPCL are all binary logical spaces, and so $\preccurlyeq_{\mathcal{F}}$ in these logics are Boolean relations by Theorem 6.8.

6.3. Generation of theories and a classification of logical spaces

The purpose of this subsection it to prove Theorem 6.10 which shows how all the theories of a logical space are generated by the given theories and use it for a classification in Definition 6.3 of logical spaces. Theorem 6.10 is also crucial to §8.

Theorem 6.9. Let $(\mathsf{A}, \mathcal{B}_{\mathsf{i}})$ be a logical space for $\mathsf{i} = 1, 2$. Then the following three conditions are equivalent.

(1) The set of the $\mathcal{B}_1$-logics is equal to that of the $\mathcal{B}_2$-logics.
(2) The set of the $\mathcal{B}_1$-theories is equal to that of the $\mathcal{B}_2$-theories.
(3) The largest $\mathcal{B}_1$-logic is equal to the largest $\mathcal{B}_2$-logic.

Under these equivalent conditions, the $\mathcal{B}_1$-core is equal to the $\mathcal{B}_2$-core.

Proof. As for the former assertion, obviously (1) implies (3). Also, Theorem 6.1 and the remark before it shows that (3) implies (2) and (2) implies (1). The latter assertion is a consequence of Theorem 6.4. □

Definition 6.2. If logical spaces $(\mathsf{A}, \mathcal{B}_1)$ and $(\mathsf{A}, \mathcal{B}_2)$ satisfy the three equivalent conditions of Theorem 6.9, we say that $(\mathsf{A}, \mathcal{B}_1)$ and $(\mathsf{A}, \mathcal{B}_2)$ are **equivalent**.

Lemma 6.1. Let $(\mathsf{A}, \mathcal{B})$ and $(\mathsf{A}, \mathcal{B}')$ be logical spaces and assume $\mathcal{B} \subseteq \mathcal{B}' \subseteq \overline{\mathcal{B}^{\cap}}$, where $\overline{\mathcal{B}^{\cap}}$ is the quasi-finitary $\cap$-closure of $\mathcal{B}$ in $\mathcal{P}\mathsf{A}$. Then $(\mathsf{A}, \mathcal{B})$ and $(\mathsf{A}, \mathcal{B}')$ are equivalent (cf. Corollary 6.10.1).

Proof. Since $\mathcal{B} \subseteq \mathcal{B}'$, every $\mathcal{B}'$-logic is a $\mathcal{B}$-logic. Let R be a $\mathcal{B}$-logic and $\mathcal{T}$ be the set of the R-theories. Then $\mathcal{B} \subseteq \mathcal{T}$, and so $\mathcal{B}' \subseteq \overline{\mathcal{B}^{\cap}} \subseteq \overline{\mathcal{T}^{\cap}} = \mathcal{T}$ by Theorems 2.7 and 4.4. Therefore R is a $\mathcal{B}'$-logic. Thus, $(\mathsf{A}, \mathcal{B})$ and $(\mathsf{A}, \mathcal{B}')$ are equivalent. □

Theorem 6.10. Let $(\mathsf{A}, \mathcal{B})$ be a logical space. Then the set of the $\mathcal{B}$-theories is equal to the quasi-finitary $\cap$-closure $\overline{\mathcal{B}^{\cap}}$ of $\mathcal{B}$ in $\mathcal{P}\mathsf{A}$.

Proof due to Matsuda. Lemma 6.1 and Theorem 6.9 show that the set of the $\mathcal{B}$-theories is equal to that of the $\overline{\mathcal{B}^{\cap}}$-theories. Therefore, we may assume $\mathcal{B} = \overline{\mathcal{B}^{\cap}}$, and need to show that every $\mathcal{B}$-theory X is super-covered by $\mathcal{B}$. Therefore, let $Y \in \mathcal{P}'X$. Define $B' = \bigcap_{Y \subseteq B \in \mathcal{B}} B$. Then since $\mathcal{B}$ is $\cap$-closed in $\mathcal{P}A$, we have $Y \subseteq B' \in \mathcal{B}$. Let $Y = \{y_1, \ldots, y_n\}$ and define $\alpha = y_1 \cdots y_n \in A^*$. Then $B' = \bigcap_{\alpha \subseteq B \in \mathcal{B}} B$, so by Theorem 6.2, every element $y \in B'$ satisfies $\alpha \, Q \, y$ for the largest $\mathcal{B}$-logic Q. Since $\alpha \subseteq X$ and X is closed under Q, we have $B' \subseteq X$, hence $Y \subseteq B' \subseteq X$ and $B' \in \mathcal{B}$. Thus X is super-covered by $\mathcal{B}$, as desired. See §9 for an alternative proof. □

Corollary 6.10.1. Two logical spaces $(A, \mathcal{B})$, $(A, \mathcal{B}')$ are equivalent iff $\overline{\mathcal{B}^{\cap}} = \overline{\mathcal{B}'^{\cap}}$.

Proof. This follows from Theorem 6.10 and Definition 6.2. □

Corollary 6.10.2. Let $(A, \mathcal{B})$ be a logical space and X be a $\mathcal{B}$-theory different from A. Then $X \subseteq \bigcup_{B \in \mathcal{B} - \{A\}} B$ (see Theorem 8.4 (2) for a generalization).

Proof. This is because $\bigcup_{X \in \overline{\mathcal{B}^{\cap}} - \{A\}} X = \bigcup_{B \in \mathcal{B} - \{A\}} B$ by Theorems 2.5 and 2.6. □

Definition 6.3. We put logical spaces $(A, \mathcal{B})$ into the following three **classes**.

- **Class 1:** $\overline{\mathcal{B}^{\cap}} = \mathcal{B}$, that is, $\mathcal{B}$ is $\cap$-closed in $\mathcal{P}A$ and quasi-finitary.
- **Class 2:** $\overline{\mathcal{B}^{\cap}} = \mathcal{B}^{\cap} \neq \mathcal{B}$, that is, $\mathcal{B}$ is not $\cap$-closed in $\mathcal{P}A$ and the $\cap$-closure $\mathcal{B}^{\cap}$ of $\mathcal{B}$ is quasi-finitary.
- **Class 3:** $\overline{\mathcal{B}^{\cap}} \neq \mathcal{B}^{\cap}$, that is, the $\cap$-closure $\mathcal{B}^{\cap}$ of $\mathcal{B}$ in $\mathcal{P}A$ is not quasi-finitary.

Since $\overline{\mathcal{B}^{\cap}} \supseteq \mathcal{B}^{\cap} \supseteq \mathcal{B}$, we have $\overline{\mathcal{B}^{\cap}} = \mathcal{B}$ iff $\overline{\mathcal{B}^{\cap}} = \mathcal{B}^{\cap} = \mathcal{B}$. Therefore, every logical space belongs to one and only one of the above classes.

Remark 6.3. A logical space $(A, \mathcal{B})$ belongs to Class 1 iff for some deduction system (R, D) on A, $\mathcal{B}$ is equal to the set of the R-theories which contain D. Corollary 6.10.1 shows that every logical space $(A, \mathcal{B})$ is equivalent to the logical space $(A, \overline{\mathcal{B}^{\cap}})$ which belongs to Class 1. Every Boolean logical space is equivalent to a binary logical space in Class 1 or 2, and a binary logical space $(A, \mathcal{F})$ belongs to Class 1 iff $\#\mathcal{F} \leq 1$.

Example 6.2. The logical spaces in PL and FPL belong to Class 2. The logical space in MPCL belongs to Class 2 or 3 according to a certain parameter, and Takaoka [13] has obtained a necessary and sufficient condition for it to belong to Class 2. See Example 8.2 for further details.

7. The completeness of deduction systems

Here we define the completeness of deduction systems and prove a fundamental theorem 7.12 on it. See Definition 8.1 for the completeness of sets.

7.1. Completeness for logical spaces

Throughout this subsection, $(\mathsf{A}, \mathcal{B})$ is a logical space, Q is the largest $\mathcal{B}$-logic on A, and C is the $\mathcal{B}$-core. Recall (5.1).

Definition 7.1. Let (R, D) be a deduction system on A. Then

- (R, D) is said to be $\mathcal{B}$-**sound** if $\mathsf{R}^{\mathsf{D}} \subseteq \mathsf{Q}$.
- (R, D) is said to be $\mathcal{B}$-**sufficient** if $\mathsf{Q} \subseteq \mathsf{R}^{\mathsf{D}}$.
- (R, D) is said to be $\mathcal{B}$-**complete** if $\mathsf{Q} = \mathsf{R}^{\mathsf{D}}$.
- (R, D) is said to be $\mathcal{B}$-**core-sound** if $[\mathsf{D}]_{\mathsf{R}} \subseteq \mathsf{C}$.
- (R, D) is said to be $\mathcal{B}$-**core-sufficient** if $\mathsf{C} \subseteq [\mathsf{D}]_{\mathsf{R}}$.
- (R, D) is said to be $\mathcal{B}$-**core-complete** if $\mathsf{C} = [\mathsf{D}]_{\mathsf{R}}$.
- (R, D) is said to be $\mathcal{B}$-**extra-complete**, if it is $\mathcal{B}$-sound and every R-theory containing D belongs to the $\cap$-closure $\mathcal{B}^{\cap}$ of $\mathcal{B}$ in $\mathcal{P}\mathsf{A}$.
- (R, D) is said to be $\mathcal{B}$-**super-complete**, if the set of the R-theories containing D is equal to $\mathcal{B}$.

The following theorems in this subsection explain meanings of the above concepts and interrelations between them.

Theorem 7.1. Let (R, D) be a deduction system on A. Then the following four conditions are equivalent.

(1) (R, D) is $\mathcal{B}$-sound.
(2) $[\mathsf{X} \cup \mathsf{D}]_{\mathsf{R}} \subseteq [\mathsf{X}]_{\mathsf{Q}}$ for every subset X of A.
(3) Every $\mathcal{B}$-theory in A is an R-theory containing D.
(4) $\mathsf{R} \subseteq \mathsf{Q}$ and $\mathsf{D} \subseteq \mathsf{C}$, that is, both R and D are $\mathcal{B}$-sound.

Also, the following three conditions are equivalent.

(5) (R, D) is $\mathcal{B}$-sufficient.
(6) $[\mathsf{X}]_{\mathsf{Q}} \subseteq [\mathsf{X} \cup \mathsf{D}]_{\mathsf{R}}$ for every subset X of A.
(7) Every R-theory in A containing D is a $\mathcal{B}$-theory.

Therefore, the following four conditions are equivalent.

(8) (R, D) is $\mathcal{B}$-complete.
(9) $[\mathsf{X}]_{\mathsf{Q}} = [\mathsf{X} \cup \mathsf{D}]_{\mathsf{R}}$ for every subset X of A.
(10) The set of the $\mathcal{B}$-theories in A is equal to the set of the R-theories in A containing D.
(11) $\mathsf{R} \subseteq \mathsf{Q}$, $\mathsf{D} \subseteq \mathsf{C}$, and $\mathsf{Q} \subseteq \mathsf{R}^{\mathsf{D}}$.

Proof. The Q is a partially latticed relation by Theorem 6.3, the $\mathcal{B}$-theories are identical with the Q-theories by Theorem 6.1, and $\mathsf{C} = \mathsf{A}_{\mathsf{Q}}$ by Theorem 6.4. Therefore, Theorem 7.1 is a consequence of Theorem 5.7. □

Theorem 7.2. Let (R, D) be a deduction system on A. If (R, D) is $\mathcal{B}$-sound, then (R, D) is $\mathcal{B}$-core-sound. If (R, D) is $\mathcal{B}$-sufficient, then (R, D) is $\mathcal{B}$-core-sufficient. Therefore if (R, D) is $\mathcal{B}$-complete, then (R, D) is $\mathcal{B}$-core-complete.

Proof. If (R,D) is $\mathcal{B}$-sound, then $[\mathsf{X}\cup\mathsf{D}]_\mathsf{R} \subseteq [\mathsf{X}]_\mathsf{Q}$ for every subset X of A by Theorem 7.1, hence in particular $[\mathsf{D}]_\mathsf{R} \subseteq [\emptyset]_\mathsf{Q}$, and since $\mathsf{C} = [\emptyset]_\mathsf{Q}$ by Theorem 6.4, we conclude that $[\mathsf{D}]_\mathsf{R} \subseteq \mathsf{C}$ holds. The rest of the proof is similar. □

Theorem 7.3. Let (R,D) be a deduction system on A. If (R,D) is $\mathcal{B}$-core-sufficient and there exists a mapping $\phi \in \mathsf{A}^* \times \mathsf{A} \to \mathsf{A}$ which satisfies

$$\alpha\,\mathsf{Q}\,y \implies \varepsilon\,\mathsf{Q}\,\phi(\alpha,y), \qquad\qquad \varepsilon\,\mathsf{R}^\mathsf{D}\,\phi(\alpha,y) \implies \alpha\,\mathsf{R}^\mathsf{D}\,y$$

for each $(\alpha,y) \in \mathsf{A}^* \times \mathsf{A}$, then (R,D) is $\mathcal{B}$-sufficient (cf. Theorem 7.11).

Proof. If an element $y \in \mathsf{A}$ satisfies $\varepsilon\,\mathsf{Q}\,y$, then $\varepsilon\,\mathsf{R}^\mathsf{D}\,y$ holds by Theorem 6.4 and the $\mathcal{B}$-core-sufficiency of (R,D). Therefore, if an element $(\alpha,y) \in \mathsf{A}^*\times\mathsf{A}$ satisfies $\alpha\,\mathsf{Q}\,y$, then $\varepsilon\,\mathsf{Q}\,\phi(\alpha,y)$, and so $\varepsilon\,\mathsf{R}^\mathsf{D}\,\phi(\alpha,y)$, hence $\alpha\,\mathsf{R}^\mathsf{D}\,y$. Thus $\mathsf{Q} \subseteq \mathsf{R}^\mathsf{D}$. □

Theorem 7.4. Let (R,D) be a deduction system on A. Then the following conditions are equivalent.

(1) (R,D) is $\mathcal{B}$-extra-complete.
(2) (R,D) is $\mathcal{B}$-complete and $(\mathsf{A},\mathcal{B})$ belongs to Class 1 or 2.

Proof. Let $\mathcal{D}$ be the set of the R-theories in A which contain D, and assume (1). Then $\mathcal{D} \subseteq \mathcal{B}^\cap \subseteq \overline{\mathcal{B}^\cap}$. In particular $\mathcal{D} \subseteq \overline{\mathcal{B}^\cap}$, so (R,D) is $\mathcal{B}$-sufficient by Theorems 6.10 and 7.1. Therefore (R,D) is $\mathcal{B}$-complete, and so $\overline{\mathcal{B}^\cap} = \mathcal{D}$ by the same theorems. Therefore $\overline{\mathcal{B}^\cap} = \mathcal{B}^\cap$, that is, $(\mathsf{A},\mathcal{B})$ belongs to Class 1 or 2. Thus (2) holds. Conversely assume (2). Then (R,D) is $\mathcal{B}$-sound, and $\mathcal{D} = \overline{\mathcal{B}^\cap} = \mathcal{B}^\cap$ by Theorems 6.10 and 7.1. Thus, (R,D) is $\mathcal{B}$-extra-complete. □

Theorem 7.5. Let (R,D) be a deduction system on A. Then the following conditions are equivalent.

(1) (R,D) is $\mathcal{B}$-super-complete.
(2) (R,D) is $\mathcal{B}$-extra-complete and $\mathcal{B}^\cap = \mathcal{B}$.
(3) (R,D) is $\mathcal{B}$-complete and $(\mathsf{A},\mathcal{B})$ belongs to Class 1.

Proof. By Theorem 7.4, (2) holds iff (R,D) is $\mathcal{B}$-complete and $\overline{\mathcal{B}^\cap} = \mathcal{B}^\cap = \mathcal{B}$. This equation means that $(\mathsf{A},\mathcal{B})$ belongs to Class 1. Thus (2) is equivalent to (3). Let $\mathcal{D}$ be the set of the R-theories in A which contain D, and assume (1). Then, since $\mathcal{D} = \mathcal{B}$, we have the following. First, R is $\mathcal{B}$-sound because it closes every element of $\mathcal{B}$. Secondly, D is $\mathcal{B}$-sound because it is contained in every element of $\mathcal{B}$ and hence in C. Thirdly, every member of $\mathcal{D}$ is a $\mathcal{B}$-theory. Therefore, (R,D) is $\mathcal{B}$-complete by Theorem 7.1. Furthermore, $\overline{\mathcal{B}^\cap} = \mathcal{D} = \mathcal{B}$ by Theorems 6.10 and 7.1, so $(\mathsf{A},\mathcal{B})$ belongs to Class 1. Thus (3) holds. Conversely assume (3). Then $\mathcal{D} = \overline{\mathcal{B}^\cap} = \mathcal{B}$ by Theorems 7.1 and 6.10, and so (R,D) is $\mathcal{B}$-super-complete. □

Theorem 7.6. The following holds for each logical space $(\mathsf{A},\mathcal{B})$.

(1) If a deduction system on A is $\mathcal{B}$-super-complete, then $(\mathsf{A},\mathcal{B})$ belongs to Class 1. Conversely, if $(\mathsf{A},\mathcal{B})$ belongs to Class 1, then every $\mathcal{B}$-complete deduction system on A is $\mathcal{B}$-super-complete.

(2) If a deduction system on A is $\mathcal{B}$-extra-complete but not $\mathcal{B}$-super-complete, then $(\mathsf{A}, \mathcal{B})$ belongs to Class 2. Conversely, if $(\mathsf{A}, \mathcal{B})$ belongs to Class 2, then every $\mathcal{B}$-complete deduction system on A is $\mathcal{B}$-extra-complete but not $\mathcal{B}$-super-complete.

(3) If a deduction system on A is $\mathcal{B}$-complete but not $\mathcal{B}$-extra-complete, then $(\mathsf{A}, \mathcal{B})$ belongs to Class 3. Conversely, if $(\mathsf{A}, \mathcal{B})$ belongs to Class 3, then no deduction system on A is $\mathcal{B}$-extra-complete.

Proof. (1) and (3) are restatements of part of Theorems 7.5 and 7.4 respectively. If a deduction system (R, D) on A is $\mathcal{B}$-extra-complete but not $\mathcal{B}$-super-complete, then (R, D) is $\mathcal{B}$-complete and $(\mathsf{A}, \mathcal{B})$ belongs to Class 1 or 2 by Theorem 7.4, and $(\mathsf{A}, \mathcal{B})$ belongs to Class 2 by Theorem 7.5. Conversely if $(\mathsf{A}, \mathcal{B})$ belongs to Class 2, then a $\mathcal{B}$-complete deduction system on A is $\mathcal{B}$-extra-complete by Theorem 7.4 but not $\mathcal{B}$-super-complete by Theorem 7.5. $\square$

Theorem 7.7. If logical spaces $(\mathsf{A}, \mathcal{B})$ and $(\mathsf{A}, \mathcal{B}')$ are equivalent to each other, then the $\mathcal{B}$-completeness and the $\mathcal{B}$-core-completeness are respectively identical with the $\mathcal{B}'$-completeness and the $\mathcal{B}'$-core-completeness, and similarly for the soundness and the sufficiency.

Proof. This is because the largest $\mathcal{B}$-logic is equal to the largest $\mathcal{B}'$-logic and the $\mathcal{B}$-core is equal to the $\mathcal{B}'$-core by Theorem 6.9. $\square$

Corollary 7.7.1. Suppose logical spaces $(\mathsf{A}, \mathcal{B})$ and $(\mathsf{A}, \mathcal{B}')$ satisfy $\mathcal{B}^{\cap} = \mathcal{B}'^{\cap}$. Then the $\mathcal{B}$-extra-completeness is identical with the $\mathcal{B}'$-extra-completeness.

Proof. Since $\overline{\mathcal{B}^{\cap}} = \overline{\mathcal{B}'^{\cap}}$, $(\mathsf{A}, \mathcal{B})$ and $(\mathsf{A}, \mathcal{B}')$ are equivalent by Corollary 6.10.1, and so the $\mathcal{B}$-soundness is identical with the $\mathcal{B}'$-soundness. Hence the above result. $\square$

7.2. Completeness for functional logical spaces

Let $(\mathsf{A}, \mathcal{F})$ be a functional logical space. We have associated it with the logical space $(\mathsf{A}, \mathcal{B}_{\mathcal{F}})$ in §6.2 and defined the $\mathcal{B}_{\mathcal{F}}$-completeness, $\mathcal{B}_{\mathcal{F}}$-soundness, $\mathcal{B}_{\mathcal{F}}$-sufficiency, and so on in §7.1. Call them the **$\mathcal{F}$-completeness, $\mathcal{F}$-soundness, $\mathcal{F}$-sufficiency**, and so on. The purpose of this subsection is to connect these concepts with the values of the functions in $\mathcal{F}$. Theorem 7.8 will help the reader understand the meanings of the concepts particularly when $\mathbb{B} = \mathbb{T}$.

Recall that, if (R, D) is a deduction system on A, then the partial deduction relation $\vDash_{\mathsf{R},\mathsf{D}}$ is equal to R^{D} and is the restriction to $\mathsf{A}^* \times \mathsf{A}$ of the deduction relation $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ defined by (5.2).

Theorem 7.8. Let $(\mathsf{A}, \mathcal{F})$ be a $\mathbb{B}$-valued functional logical space and (R, D) be a deduction system on A. Then the following holds.

(1) (R, D) is $\mathcal{F}$-sound iff $\vDash_{\mathsf{R},\mathsf{D}}$ is contained in $\vDash_{\mathcal{F}}$, that is, iff it satisfies the following condition for all $(f, a) \in \mathcal{F} \times \mathbb{B}$:

$$fx_1 \geq a, \ \ldots, \ fx_n \geq a, \ x_1 \cdots x_n \vDash_{\mathsf{R},\mathsf{D}} y \implies fy \geq a.$$

(2) (R,D) is $\mathcal{F}$-sufficient iff $\vDash_{\mathcal{F}}$ is contained in $\vDash_{\mathsf{R},\mathsf{D}}$, that is, iff it satisfies

$$x_1 \cdots x_n \nvDash_{\mathsf{R},\mathsf{D}} y \implies fx_1 \geq a, \ \ldots, \ fx_n \geq a \text{ but } fy \ngeq a \text{ for some } (f,a) \in \mathcal{F} \times \mathbb{B}.$$

(3) (R,D) is $\mathcal{F}$-complete iff $\vDash_{\mathsf{R},\mathsf{D}}$ is equal to $\vDash_{\mathcal{F}}$, that is, iff it satisfies

$$x_1 \cdots x_n \nvDash_{\mathsf{R},\mathsf{D}} y \iff fx_1 \geq a, \ \ldots, \ fx_n \geq a \text{ but } fy \ngeq a \text{ for some } (f,a) \in \mathcal{F} \times \mathbb{B}.$$

Proof. This is a restatement of part of Definition 7.1 by Theorem 6.5. □

Corollary 7.8.1. Let $(A,\mathcal{F})$ be a functional logical space and (R,D) be a deduction system on A. Then the following holds.

(1) If $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is contained in $\preccurlyeq_{\mathcal{F}}$, then (R,D) is $\mathcal{F}$-sound.
(2) If $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ contains $\preccurlyeq_{\mathcal{F}}$, then (R,D) is $\mathcal{F}$-sufficient.
(3) If $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is equal to $\preccurlyeq_{\mathcal{F}}$, then (R,D) is $\mathcal{F}$-complete.

Proof. Since the restrictions of $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ and $\preccurlyeq_{\mathcal{F}}$ to $A^* \times A$ are equal to $\vDash_{\mathsf{R},\mathsf{D}}$ and $\vDash_{\mathcal{F}}$ respectively, (1)–(3) are consequences of Theorem 7.8. □

Theorem 7.9. Let $(A,\mathcal{F})$ be a functional logical space, assume that $\preccurlyeq_{\mathcal{F}}$ is the largest latticed extension of $\vDash_{\mathcal{F}}$, and let (R,D) be a deduction system on A. Then, (R,D) is $\mathcal{F}$-complete iff $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is equal to $\preccurlyeq_{\mathcal{F}}$.

Proof. Assume that(R,D) is $\mathcal{F}$-complete. Then $\vDash_{\mathsf{R},\mathsf{D}}$ is equal to $\vDash_{\mathcal{F}}$ by Theorem 7.8. Since $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is the largest latticed extension of $\vDash_{\mathsf{R},\mathsf{D}}$ by Theorem 5.2, while $\preccurlyeq_{\mathcal{F}}$ is the largest latticed extension of $\vDash_{\mathcal{F}}$ by our assumption, we conclude that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is equal to $\preccurlyeq_{\mathcal{F}}$. The converse has been proved in Corollary 7.8.1. □

7.3. Boolean features of completeness

In view of Example 6.1, here we study the completeness on Boolean logical spaces.

Theorem 7.10. Let $(A,\mathcal{F})$ be a Boolean logical space with respect to operations $\wedge, \vee, \lozenge, \Rightarrow$ on A and (R,D) be a deduction system on A. Then (R,D) is $\mathcal{F}$-complete iff $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is equal to $\preccurlyeq_{\mathcal{F}}$. Therefore, (R,D) is $\mathcal{F}$-complete only if $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is a Boolean relation with respect to $\wedge, \vee, \lozenge, \Rightarrow$.

Proof. Since $\preccurlyeq_{\mathcal{F}}$ is the largest latticed extension of $\vDash_{\mathcal{F}}$ by Theorem 6.8, the former assertion holds by Theorem 7.9. Since $\preccurlyeq_{\mathcal{F}}$ is a Boolean relation by Theorem 6.8, the latter assertion follows from the former. □

Example 7.1. Let $(A,\mathcal{F})$ be a Boolean logical space with respect to operations $\wedge, \vee, \lozenge, \Rightarrow$ on A. Let $\wp, \&$ be logics on A defined by (5.3) and define $\mathsf{R} = \wp \cup \&$. Let D be the set of the elements of A listed in Corollary 5.9.1. Then, $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is a Boolean relation with respect to $\wedge, \vee, \lozenge, \Rightarrow$. The deduction system (R,D) is also $\mathcal{F}$-sound. As for the soundness of R, since $\preccurlyeq_{\mathcal{F}}$ is a Boolean relation, both $\wp$ and $\&$ are contained in $\vDash_{\mathcal{F}}$ by Theorem 3.14 and Definition 3.3. Therefore both $\wp$ and $\&$ are $\mathcal{F}$-sound by Theorem 6.6. If $(A,\mathcal{F})$ is the Boolean logical space in PL, then

(R, D) is $\mathcal{F}$-complete. As for FPL, MPCL, and PPCL, by virtue of Corollary 7.11.1, certain $\mathcal{F}$-complete deduction systems $(\wp, D')$ with $D \subseteq D'$ have been found.

Theorem 7.11. Let $(A, \mathcal{F})$ be a Boolean logical space with respect to operations $\wedge, \vee, \Diamond, \Rightarrow$ on A and (R, D) be an $\mathcal{F}$-core-sufficient deduction system on A and assume that R contains the modus ponens $\dfrac{x \quad x \Rightarrow y}{y}$. Then (R, D) is $\mathcal{F}$-sufficient.

Proof. Define the mapping $\phi \in A^* \times A \to A$ by

$$\phi(x_1 \cdots x_n, y) = x_n \Rightarrow (\cdots \Rightarrow (x_2 \Rightarrow (x_1 \Rightarrow y)) \cdots).$$

Then since $\preccurlyeq_{\mathcal{F}}$ is a Boolean relation by Theorem 6.8 and $\vDash_{\mathcal{F}}$ is the restriction of $\preccurlyeq_{\mathcal{F}}$ to $A^* \times A$, Theorem 3.14 shows that

$$x_1 \cdots x_n \vDash_{\mathcal{F}} y \iff \varepsilon \vDash_{\mathcal{F}} \phi(x_1 \cdots x_n, y)$$

holds. In particular, ϕ satisfies "$\alpha \vDash_{\mathcal{F}} y \implies \varepsilon \vDash_{\mathcal{F}} \phi(\alpha, y)$" for every element $(\alpha, y) \in A^* \times A$. Assume $\varepsilon \vDash_{R,D} \phi(x_1 \cdots x_n, y)$. Then

$$x_n \Rightarrow (\cdots \Rightarrow (x_2 \Rightarrow (x_1 \Rightarrow y)) \cdots) \in [D]_R \subseteq [\{x_1, \ldots, x_n\} \cup D]_R.$$

Since R contains the modus ponens, it inductively follows that

$$x_{n-i} \Rightarrow (\cdots \Rightarrow (x_2 \Rightarrow (x_1 \Rightarrow y)) \cdots) \in [\{x_1, \ldots, x_n\} \cup D]_R$$

holds for $i = 1, \ldots, n$. Hence $y \in [\{x_1, \ldots, x_n\} \cup D]_R$, that is, $x_1 \cdots x_n \vDash_{R,D} y$. Thus ϕ satisfies "$\varepsilon \vDash_{R,D} \phi(\alpha, y) \implies \alpha \vDash_{R,D} y$" for every element $(\alpha, y) \in A^* \times A$. The result now follows from Theorems 6.5 and 7.3. $\square$

Corollary 7.11.1. Let $(A, \mathcal{F})$ be a Boolean logical space with respect to operations $\wedge, \vee, \Diamond, \Rightarrow$ on A. Then the deduction system $(\wp, C)$ on A consisting of the modus ponens $\wp = \dfrac{x \quad x \Rightarrow y}{y}$ and the $\mathcal{F}$-core C is $\mathcal{F}$-complete.

Proof. $(\wp, C)$ is $\mathcal{F}$-sound by Example 7.1 and $\mathcal{F}$-sufficient by Theorem 7.11. $\square$

7.4. Fundamental theorem of completeness

Let $(A, \mathcal{F})$ be a functional logical space. Then as in Example 5.2, we define

$$\vec{A} = A^* \times A^*,$$

denote the elements $(\alpha, \beta) \in \vec{A}$ by $\alpha \to \beta$, and call them **sequents**. Also we define

$$\begin{aligned} \vec{C}_{\mathcal{F}} &= \{\alpha \to \beta \in \vec{A} \mid \alpha \preccurlyeq_{\mathcal{F}} \beta\}, \\ \vec{A}_f &= \{\alpha \to \beta \in \vec{A} \mid \alpha \preccurlyeq_f \beta\} \qquad\qquad (f \in \mathcal{F}), \\ \vec{\mathcal{F}} &= \{\vec{A}_f \mid f \in \mathcal{F}\}. \end{aligned}$$

Then $(\vec{A}, \vec{\mathcal{F}})$ is a logical space, which we call the **sequential logical space** associated with $(A, \mathcal{F})$. The $\vec{\mathcal{F}}$-core of $\vec{A}$ is equal to $\vec{C}_{\mathcal{F}}$, because (6.1) means $\vec{C}_{\mathcal{F}} = \bigcap_{f \in \mathcal{F}} \vec{A}_f$.

Recall from §5.2 that deduction systems on $\vec{A}$ are also called generational laws on the relations on A^*. Less importantly, they are also said to be of Gentzen style, while deduction systems on A are said to be of Hilbert style.

Definition 7.2. A deduction system $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ on $\vec{\mathsf{A}}$ is called a **characteristic law** of $(\mathsf{A}, \mathcal{F})$ if $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ is $\vec{\mathcal{F}}$-core-complete, that is, if $\vec{\mathsf{C}}_{\mathcal{F}} = [\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$. In other words, a characteristic law of $(\mathsf{A}, \mathcal{F})$ is a generational law $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ on the relations on A^* which makes $\preccurlyeq_{\mathcal{F}}$ the smallest $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$-relation.

Remark 7.1. $(\mathsf{A}, \mathcal{F})$ may have several "inequivalent" characteristic laws other than the trivial one $(\emptyset, \vec{\mathsf{C}}_{\mathcal{F}})$. For instance, the Boolean law and weakly Boolean law are both characteristic laws of PL and inequivalent.

Theorem 7.12 (fundamental theorem of completeness). Let $(\mathsf{A}, \mathcal{F})$ be a functional logical space and (R, D) be an $\mathcal{F}$-sound deduction system on A. Assume that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies a characteristic law $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ of $(\mathsf{A}, \mathcal{F})$. Then (R, D) is $\mathcal{F}$-complete.

Proof. We more generally prove that (R, D) is $\mathcal{F}$-sufficient provided that, for some deduction system $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ on $\vec{\mathsf{A}}$, (1) $\vec{\mathsf{C}}_{\mathcal{F}} \subseteq [\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$ and (2) $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the generational law $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$. Define $\vec{\mathsf{A}}_{\mathsf{R},\mathsf{D}} = \{\alpha \to \beta \in \vec{\mathsf{A}} \mid \alpha \preccurlyeq_{\mathsf{R},\mathsf{D}} \beta\}$. Then (2) means that $\vec{\mathsf{A}}_{\mathsf{R},\mathsf{D}}$ is closed under $\vec{\mathsf{R}}$ and contains $\vec{\mathsf{D}}$, hence $[\vec{\mathsf{D}}]_{\vec{\mathsf{R}}} \subseteq \vec{\mathsf{A}}_{\mathsf{R},\mathsf{D}}$. Therefore $\vec{\mathsf{C}}_{\mathcal{F}} \subseteq \vec{\mathsf{A}}_{\mathsf{R},\mathsf{D}}$ by (1), which means that $\preccurlyeq_{\mathcal{F}}$ is contained in $\preccurlyeq_{\mathsf{R},\mathsf{D}}$. Therefore (R, D) is $\mathcal{F}$-sufficient by Corollary 7.8.1. $\square$

Remark 7.2. The following partial converse of Theorem 7.12 holds. Let $(\mathsf{A}, \mathcal{F})$ be a functional logical space and (R, D) be an $\mathcal{F}$-complete deduction system on A. Assume that $\preccurlyeq_{\mathcal{F}}$ is the largest latticed extension of $\vDash_{\mathcal{F}}$ (cf. Theorem 6.8). Then $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies every characteristic law $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ of $(\mathsf{A}, \mathcal{F})$.

Remark 7.3. Theorem 7.12 and Remark 7.2 give the following practically necessary and sufficient program for finding an $\mathcal{F}$-complete deduction system (R, D) on A.

- **1st step:** Find a characteristic law $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ of $(\mathsf{A}, \mathcal{F})$, that is, find a $\vec{\mathcal{F}}$-core-complete deduction system $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ on $\vec{\mathsf{A}}$.
- **2nd step:** Find an $\mathcal{F}$-sound deduction system (R, D) on A whose deduction relation $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$.

Corollary 5.9.1 is relevant to the 2nd step for all of PL, FPL, MPCL, and PPCL, because $\preccurlyeq_{\mathcal{F}}$ in these logics are Boolean relations by Example 6.1, and the Boolean law has naturally been extended to their characteristic laws.

7.5. Finding characteristic laws by means of Dedekind cuts

Here we extract essence for the 1st step of Remark 7.3 from Henkin's proof [7] of Gödel's completeness theorem [3].

Let $(\mathsf{A}, \mathcal{F})$ be a $\mathbb{T}$-valued functional logical space and $(\mathsf{X}, \mathsf{Y}) \in \mathcal{P}\mathsf{A} \times \mathcal{P}\mathsf{A}$. Then an element $\mathsf{f} \in \mathcal{F}$ is called an **$\mathcal{F}$-model** of (X, Y) if $\mathsf{X} \subseteq \mathsf{f}^{-1}1 \neq \mathsf{A}$ and $\mathsf{Y} \subseteq \mathsf{f}^{-1}0$.

Let $\preccurlyeq$ be a relation on A^*. Then an element $(\mathsf{X}, \mathsf{Y}) \in \mathcal{P}\mathsf{A} \times \mathcal{P}\mathsf{A}$ is called a **cut** or a **Dedekind cut** of A by $\preccurlyeq$ if, for each $(\alpha, \beta) \in \mathsf{A}^* \times \mathsf{A}^*$ with $\alpha \subseteq \mathsf{X}$ and $\beta \subseteq \mathsf{Y}$, $\alpha \not\preccurlyeq \beta$ holds. The cut (X, Y) is said to be **finite** if $(\mathsf{X}, \mathsf{Y}) \in \mathcal{P}'\mathsf{A} \times \mathcal{P}'\mathsf{A}$.

Lemma 7.1. Let $(\mathsf{A}, \mathcal{F})$ be a $\mathbb{T}$-valued functional logical space and $\preccurlyeq$ be a relation on A^* satisfying the weakening law, contraction law, and exchange law. Assume that every finite cut of A by $\preccurlyeq$ has an $\mathcal{F}$-model. Then $\preccurlyeq_{\mathcal{F}}$ is contained in $\preccurlyeq$.

Proof. Suppose an element $(\alpha, \beta) \in \mathsf{A}^* \times \mathsf{A}^*$ satisfies $\alpha \not\preccurlyeq \beta$. Then since $\preccurlyeq$ satisfies the weakening law, contraction law, and exchange law, it follows that (α, β) is a finite cut of A by $\preccurlyeq$, and so there exists an element $\mathsf{f} \in \mathcal{F}$ such that $\alpha \subseteq \mathsf{f}^{-1}1$ and $\beta \subseteq \mathsf{f}^{-1}0$, hence $\alpha \not\preccurlyeq_{\mathsf{f}} \beta$. Therefore $\alpha \not\preccurlyeq_{\mathcal{F}} \beta$. □

Theorem 7.13. Let $(\mathsf{A}, \mathcal{F})$ be a $\mathbb{T}$-valued functional logical space and $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ be a generational law on the relations on A^*. Assume the following:

(1) The $\mathcal{F}$-validity relation $\preccurlyeq_{\mathcal{F}}$ satisfies $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$.

(2) Every finite cut of A by every $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$-relation contained in $\preccurlyeq_{\mathcal{F}}$ has an $\mathcal{F}$-model.

Then the law $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ together with the weakening law, contraction law, and exchange law forms a characteristic law of $(\mathsf{A}, \mathcal{F})$.

Proof. As suggested by Example 5.2, the weakening law, contraction law, and exchange law are generational laws of the form $(\vec{\mathsf{R}}_w, \emptyset)$, $(\vec{\mathsf{R}}_c, \emptyset)$, $(\vec{\mathsf{R}}_e, \emptyset)$, respectively. Define $\vec{\mathsf{S}} = \vec{\mathsf{R}} \cup \vec{\mathsf{R}}_w \cup \vec{\mathsf{R}}_c \cup \vec{\mathsf{R}}_e$. The theorem asserts that the generational law $(\vec{\mathsf{S}}, \vec{\mathsf{D}})$ is a characteristic law of $(\mathsf{A}, \mathcal{F})$. A proof of this is as follows. The condition (1) and Theorem 6.7 show that $\preccurlyeq_{\mathcal{F}}$ satisfies $(\vec{\mathsf{S}}, \vec{\mathsf{D}})$. Regard $[\vec{\mathsf{D}}]_{\vec{\mathsf{S}}}$ as a relation on A^*, and denote it by $\preccurlyeq$. Then $\preccurlyeq$ is the smallest $(\vec{\mathsf{S}}, \vec{\mathsf{D}})$-relation. Therefore, $\preccurlyeq$ is a $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$-relation contained in $\preccurlyeq_{\mathcal{F}}$, and $\preccurlyeq$ satisfies the weakening law, contraction law, and exchange law. The condition (2) and Lemma 7.1 now show that $\preccurlyeq_{\mathcal{F}}$ is contained in $\preccurlyeq$. Thus $\preccurlyeq_{\mathcal{F}}$ is equal to $\preccurlyeq$, that is, $\vec{\mathsf{C}}_{\mathcal{F}} = [\vec{\mathsf{D}}]_{\vec{\mathsf{S}}}$. □

Remark 7.4. Let $(\mathsf{A}, \mathcal{F})$ be a $\mathbb{T}$-valued functional logical space with $1 \notin \mathcal{F}$ and $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ be its characteristic law. Then every finite cut of A by every $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$-relation on A^* has an $\mathcal{F}$-model. Therefore, Theorem 7.13 also gives practically necessary and sufficient program for finding a characteristic law of $(\mathsf{A}, \mathcal{F})$.

Example 7.2. Suppose $(\mathsf{A}, \mathcal{F})$ is the functional logical space in FPL, and let $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ be the well-known Gentzen style deduction system for FPL. Then the conditions (1) (2) of Theorem 7.13 are satisfied, provided that there are infinitely many variables. Therefore by Theorem 7.13, $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ is a characteristic law of $(\mathsf{A}, \mathcal{F})$. Thus if (R, D) is an $\mathcal{F}$-sound deduction system on A and $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$, then (R, D) is $\mathcal{F}$-complete by Theorem 7.12. Furthermore, every (not necessarily finite) cut of A by $\preccurlyeq_{\mathcal{F}}$ has an $\mathcal{F}$-model, irrespective of the cardinality of the variables. If X is an $\mathcal{F}$-consistent subset of A in the sense of Definition 8.1 below, then $(\mathsf{X}, \emptyset)$ is a cut of A by $\preccurlyeq_{\mathcal{F}}$, and so X has an $\mathcal{F}$-model in the sense of Definition 8.2 below. Takaoka [13] has proved similar theorems for MPCL.

Remark 7.5. Suppose $(\mathsf{A}, \mathcal{F})$ is a binary logical space. Then a subset X of A is $\mathcal{F}$-consistent iff $(\mathsf{X}, \emptyset)$ is a cut of A by $\preccurlyeq_{\mathcal{F}}$. Also, an element $\mathsf{f} \in \mathcal{F}$ is an $\mathcal{F}$-model of $(\mathsf{X}, \emptyset)$ iff f is an $\mathcal{F}$-model of X in the sense of Definition 8.2 below. Therefore, verification of Theorem 7.13 (2) unites with the problem of how to construct $\mathcal{F}$-models of $\mathcal{F}$-consistent subsets of A.

8. Classes and existence of models

In §7.1, we observed interrelations between the completeness of deduction systems on logical spaces and the classification in Definition 6.3 of the logical spaces. In this section, we observe that the completeness and models of consistent sets are related through the classification. The main results are Theorems 8.8 and 8.9.

Throughout the section, we let $(A, \mathcal{B})$ be a logical space, and Q be the largest $\mathcal{B}$-logic on A.

Definition 8.1. A subset X of A is said to be **$\mathcal{B}$-consistent** if $[X]_Q \neq A$.

In the rest of the section, we let $\mathcal{C}$ be the set of the $\mathcal{B}$-consistent subsets of A, $\mathcal{D}$ be the set of the maximal elements of the ordered set $(\mathcal{C}, \subseteq)$, and $\mathcal{E}$ be the set $\{X \in \mathcal{P}A \mid [X]_Q \in \mathcal{D}\}$. We call elements of $\mathcal{E}$ the **$\mathcal{B}$-complete subsets** of A.

Theorem 8.1. A finite subset $\{x_1, \ldots, x_n\}$ of A is $\mathcal{B}$-inconsistent iff every element $y \in A$ satisfies $x_1 \cdots x_n \, Q \, y$.

Proof. This is a direct consequence of Theorem 6.3. □

Theorem 8.2. Assume that there exists a $\mathcal{B}$-inconsistent finite subset $\{x_1, \ldots, x_n\}$ $(n \geq 1)$ of A. Then the following conditions on a subset X of A are equivalent.

(1) X is $\mathcal{B}$-inconsistent.
(2) $x_1, \ldots, x_n \in [X]_Q$.
(3) There exists an element $\alpha \in A^*$ such that $\alpha \subseteq X$ and $\alpha \, Q \, x_i$ $(i = 1, \ldots, n)$.
(4) There exists an element $\alpha \in A^*$ such that $\alpha \subseteq X$ and $\alpha \, Q \, y$ for all $y \in A$.
(5) There exists a $\mathcal{B}$-inconsistent finite subset of X.

Therefore $\mathcal{C}$ is a finitary subset of $\mathcal{P}A$ (see Theorem 8.4 (4) for further details).

Proof. The final remark follows from Theorem 2.1 and equivalence of (1) to (5). If (1) holds, then since $[X]_Q = A$, (2) holds. Assume (2). Then Theorem 6.3 shows that, for each $i = 1, \ldots, n$, there exists an element $\alpha_i \in A^*$ such that $\alpha_i \subseteq X$ and $\alpha_i \, Q \, x_i$. Theorem 6.3 also shows that Q satisfies the partial weakening law and partial exchange law. Therefore, (3) holds with $\alpha = \alpha_1 \cdots \alpha_n$. Assume (3) and let $y \in A$. Then $\alpha \, Q \, x_i$ $(i = 1, \ldots, n)$ and $x_1 \cdots x_n \, Q \, y$ by Theorem 8.1. By repeated application of the partial cut law, partial exchange law, and partial contraction law to the above $n + 1$ Q-relations, we have $\alpha \, Q \, y$. Thus (4) holds. If (4) holds, then α is $\mathcal{B}$-inconsistent by Theorem 8.1, and so (5) holds. If (5) holds, then since every subset Y of X satisfies $[Y]_Q \subseteq [X]_Q$ by Theorem 4.4, (1) holds. □

Theorem 8.3. The following holds on the Q-closures of subsets of A.

(1) $[Y]_Q = \bigcap_{Y \subseteq B \in \mathcal{B}} B$ for each $Y \in \mathcal{P}'A$.
(2) $(A, \mathcal{B})$ belongs to Class 1 or 2 iff $[X]_Q = \bigcap_{X \subseteq B \in \mathcal{B}} B$ for each $X \in \mathcal{P}A$.

Proof. Let $Y = \{y_1, \ldots, y_n\}$ and define $\alpha = y_1 \cdots y_n \in A^*$. Then $[Y]_Q = [\alpha]_Q = \{y \in A \mid \alpha\, Q\, y\} = \bigcap_{\alpha \subseteq B \in \mathcal{B}} B = \bigcap_{Y \subseteq B \in \mathcal{B}} B$ by Theorems 6.3 and 6.2. Alternatively, $[Y]_Q = \bigcap_{Y \subseteq X \in \overline{\mathcal{B}^\cap}} X = \bigcap_{Y \subseteq X \in \mathcal{B}^\cap} X = \bigcap_{Y \subseteq B \in \mathcal{B}} B$ by Theorems 6.1, 6.10, 2.6, and 2.5. If $(A, \mathcal{B})$ belongs to Class 1 or 2, then $\overline{\mathcal{B}^\cap} = \mathcal{B}^\cap$, so $[X]_Q = \bigcap_{X \subseteq Y \in \overline{\mathcal{B}^\cap}} Y = \bigcap_{X \subseteq Y \in \mathcal{B}^\cap} Y = \bigcap_{X \subseteq B \in \mathcal{B}} B$ for each $X \in \mathcal{P}A$ by Theorems 6.1, 6.10, and 2.5. Conversely, if $[X]_Q = \bigcap_{X \subseteq B \in \mathcal{B}} B$ for each $X \in \mathcal{P}A$, then for each $X \in \overline{\mathcal{B}^\cap}$, we have $X = [X]_Q = \bigcap_{X \subseteq B \in \mathcal{B}} B \in \mathcal{B}^\cap$ by Theorem 6.10, and therefore $\overline{\mathcal{B}^\cap} = \mathcal{B}^\cap$. $\square$

Definition 8.2. A **$\mathcal{B}$-model** for a subset X of A is a set $B \in \mathcal{B} - \{A\}$ containing X.

Therefore, the subset X has a $\mathcal{B}$-model iff X belongs to the downward closure $\overleftarrow{\mathcal{B} - \{A\}}$ of $\mathcal{B} - \{A\}$ and iff $\bigcap_{X \subseteq B \in \mathcal{B}} B \neq A$.

Theorem 8.4. The following holds on $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$.

(1) $\overleftarrow{\mathcal{B} - \{A\}} \subseteq \mathcal{C} = \overleftarrow{\overline{\mathcal{B}^\cap} - \{A\}} \subseteq \overline{\overleftarrow{\mathcal{B} - \{A\}}}$, that is, a subset of A is $\mathcal{B}$-consistent, if it has a $\mathcal{B}$-model, iff it is contained in a non-trivial $\mathcal{B}$-theory in A, and only if every finite subset of it has a $\mathcal{B}$-model.

(2) $\mathcal{P}'A \cap \mathcal{C} = \mathcal{P}'A \cap \overleftarrow{\mathcal{B} - \{A\}}$, that is, a finite subset of A is $\mathcal{B}$-consistent iff it has a $\mathcal{B}$-model.

(3) If $(A, \mathcal{B})$ belongs to Class 1 or 2, then $\mathcal{C} = \overleftarrow{\mathcal{B} - \{A\}}$, that is, a subset of A is $\mathcal{B}$-consistent iff it has a $\mathcal{B}$-model.

(4) If there exists a $\mathcal{B}$-inconsistent finite subset of A, then $\mathcal{C} = \overline{\overleftarrow{\mathcal{B} - \{A\}}} = \overleftarrow{\mathcal{D}}$, that is, a subset X of A is $\mathcal{B}$-consistent iff every finite subset of X has a $\mathcal{B}$-model, and iff X is contained in a maximal $\mathcal{B}$-consistent subset of A.

(5) $\mathcal{D}$ is equal to the set of the maximal elements of the ordered set $(\overline{\mathcal{B}^\cap} - \{A\}, \subseteq)$, that is, a subset of A is a maximal $\mathcal{B}$-consistent subset of A iff it is a maximal non-trivial $\mathcal{B}$-theory in A.

(6) An element $X \in \mathcal{P}A$ belongs to $\mathcal{C}$ iff $[X]_Q \in \mathcal{C}$. Also if $X \in \mathcal{D}$, then $X = [X]_Q$.

(7) $\mathcal{D} \subseteq \mathcal{E} \subseteq \mathcal{C}$, that is, a subset of A is $\mathcal{B}$-complete, if it is maximal $\mathcal{B}$-consistent, and only if A is $\mathcal{B}$-consistent.

(8) If $X \in \mathcal{E}$, then X has at most one $\mathcal{B}$-model, which is equal to $[X]_Q$ if any.

Proof. If $X \in \overleftarrow{\overline{\mathcal{B}^\cap} - \{A\}}$, then there exists a set $Y \in \overline{\mathcal{B}^\cap} - \{A\}$ such that $X \subseteq Y$, and since $[X]_Q \subseteq Y \neq A$ by Theorem 6.10, we have $X \in \mathcal{C}$. Thus $\overleftarrow{\mathcal{B} - \{A\}} \subseteq \overleftarrow{\overline{\mathcal{B}^\cap} - \{A\}} \subseteq \mathcal{C}$. Conversely if $X \in \mathcal{C}$, then $X \subseteq [X]_Q \in \overline{\mathcal{B}^\cap} - \{A\}$ by Theorems 6.1 and 6.10, hence $X \in \overleftarrow{\overline{\mathcal{B}^\cap} - \{A\}}$. Thus $\mathcal{C} = \overleftarrow{\overline{\mathcal{B}^\cap} - \{A\}}$. Consequently, $\mathcal{C}$ is downward. Suppose $X \in \mathcal{C}$ and $Y \in \mathcal{P}'X$. Then $\bigcap_{Y \subseteq B \in \mathcal{B}} B = [Y]_Q \neq A$ by Theorem 8.3, and so $Y \in \overleftarrow{\mathcal{B} - \{A\}}$. Therefore $\mathcal{C} \subseteq \overline{\overleftarrow{\mathcal{B} - \{A\}}}$ by Theorem 2.9. Thus (1) holds. (2) is a consequence of (1) and Theorem 2.6. Suppose $(A, \mathcal{B})$ belongs to Class 1 or 2. Then each element $X \in \mathcal{C}$ satisfies $A \neq [X]_Q = \bigcap_{X \subseteq B \in \mathcal{B}} B$ by Theorem 8.3, and so $X \in \overleftarrow{\mathcal{B} - \{A\}}$. This together with (1) implies $\mathcal{C} = \overleftarrow{\mathcal{B} - \{A\}}$. Thus (3) holds. In order to prove (4), assume that there exists a $\mathcal{B}$-inconsistent finite subset of A. Then $\mathcal{C}$ is finitary by

Theorem 8.2. Therefore, $\mathcal{C} = \overleftarrow{\overline{\mathcal{B} - \{A\}}}$ by (1) and Theorem 2.9. Also, the ordered set $(\mathcal{C}, \subseteq)$ is inductive by Theorem 2.2 and a remark about Definition 2.2, and so $\mathcal{C} \subseteq \overleftarrow{\mathcal{D}}$ by Zorn's lemma. Since $\mathcal{C}$ is downward by (1), conversely $\overleftarrow{\mathcal{D}} \subseteq \mathcal{C}$. Thus (4) holds. (5) is a consequence of (1). (6) holds because $X \subseteq [X]_Q = \big[[X]_Q\big]_Q$. If $X \in \mathcal{D}$, then $[X]_Q = X \in \mathcal{D}$ by (6), and so $X \in \mathcal{E}$. If $X \in \mathcal{E}$, then $X \subseteq [X]_Q \in \mathcal{D} \subseteq \mathcal{C}$, and so $X \in \mathcal{C}$ because $\mathcal{C}$ is downward. Thus (7) holds. If $X \in \mathcal{E}$ and $X \subseteq B \in \mathcal{B} - \{A\}$, then $\mathcal{D} \ni [X]_Q \subseteq B \in \mathcal{C}$, hence $B = [X]_Q$. Thus (8) holds. □

Definition 8.3. A $\mathcal{B}$**-negation** is a unary operation $x^\lozenge$ on A which satisfies

$$x^\lozenge \in B \iff x \notin B \qquad \text{(B-negation law)}$$

for all $B \in \mathcal{B} - \{A\}$. A $\mathcal{B}$**-complement** is a unary operation $x^\lozenge$ on A which satisfies

$$xx^\lozenge \, Q \, y, \qquad \text{(contradiction law)}$$

$$x\alpha \, Q \, y, \; x^\lozenge\alpha \, Q \, y \implies \alpha \, Q \, y. \qquad \text{(excluded middle law)}$$

Example 8.1. Boolean logical spaces $(A, \mathcal{F})$ have an $\mathcal{F}$-complement by Theorem 3.13, because Q is equal to $\vDash_\mathcal{F}$ by Theorem 6.5, and $\preccurlyeq_\mathcal{F}$ is Boolean by Theorem 6.8. The $\mathcal{F}$-complement becomes an $\mathcal{F}$-negation iff $fA \subseteq \{0, 1\}$ for all $f \in \mathcal{F}$.

Theorem 8.5. Let $x^\lozenge$ be a unary operation on A. Then the following holds.

(1) The $\lozenge$ satisfies the contradiction law iff $\{x, x^\lozenge\} \notin \mathcal{C}$ for each element $x \in A$.
(2) The $\lozenge$ satisfies the excluded middle law iff it satisfies the law

$$\big[\{x\} \cup X\big]_Q \cap \big[\{x^\lozenge\} \cup X\big]_Q \subseteq [X]_Q, \tag{8.1}$$

and iff it satisfies this law for $X \in \mathcal{P}'A$.

Proof. (1) is a consequence of Theorem 8.1. Suppose $y \in \big[\{x\} \cup X\big]_Q \cap \big[\{x^\lozenge\} \cup X\big]_Q$. Then Theorem 6.3 shows that there exists an element $\alpha \in A^*$ such that $\alpha \subseteq X$, $x\alpha \, Q \, y$ and $x^\lozenge\alpha \, Q \, y$, and so the excluded middle law yields $\alpha \, Q \, y$, hence $y \in [X]_Q$. Suppose $x\alpha \, Q \, y$ and $x^\lozenge\alpha \, Q \, y$. Then $y \in \big[\{x\} \cup \alpha\big]_Q \cap \big[\{x^\lozenge\} \cup \alpha\big]_Q$, and so (8.1) for $X \in \mathcal{P}'A$ yields $y \in [\alpha]_Q$, hence $\alpha \, Q \, y$ by Theorem 6.3. □

Corollary 8.5.1. Let $X \subseteq A$ and $y \in A$. Suppose a $\mathcal{B}$-complement $\lozenge$ on A exists. Then in order that $X \notin \mathcal{C}$ holds, either of the following conditions is necessary and sufficient.

(1) $y, y^\lozenge \in [X]_Q$.
(2) There exists an element $\alpha \in A^*$ such that $\alpha \subseteq X$, $\alpha \, Q \, y$ and $\alpha \, Q \, y^\lozenge$.

Proof. This is a consequence of Theorems 8.5 and 8.2. □

Theorem 8.6. Let $X \subseteq A$. Suppose a $\mathcal{B}$-complement $\lozenge$ on A exists. Then in order that $X \in \mathcal{D}$ holds, either of the following conditions is necessary and sufficient.

(1) $[X]_Q = X$ and $\lozenge$ satisfies the X-negation law "$x^\lozenge \in X \iff x \notin X$."
(2) X belongs to $\mathcal{C}$ and satisfies $x \in X$ or $x^\lozenge \in X$ for each element $x \in A$.

Proof. Assume $X \in \mathcal{D}$. Then $[X]_Q = X$ by Theorem 8.4. Let $x \in A$. Then $\{x, x^\Diamond\} \notin \mathcal{C}$ by Theorem 8.5, and so $\{x, x^\Diamond\} \nsubseteq X$, hence if $x^\Diamond \in X$ then $x \notin X$. In order to prove the converse, assume $x \notin X$. Then $\{x\} \cup X \notin \mathcal{C}$, that is, $[\{x\} \cup X]_Q = A$, and so $x^\Diamond \in [X]_Q = X$ by (8.1). Thus (1) holds. Assume (1). Then since $A \neq \emptyset$, the X-negation law yields $X \neq A$, and so (2) holds. Assume (2). Then if $X \subset Y \subseteq A$, any element $y \in Y - X$ satisfies $y^\Diamond \in X \subseteq Y$, and so $\{y, y^\Diamond\} \subseteq Y$, hence $Y \notin \mathcal{C}$ by Theorem 8.5. Thus $X \in \mathcal{D}$ holds. □

Corollary 8.6.1. Let $X \subseteq A$. Suppose a $\mathcal{B}$-complement $\Diamond$ on A exists. Then in order that $X \in \mathcal{E}$ holds, either of the following conditions is necessary and sufficient.

(1) $\Diamond$ satisfies the $[X]_Q$-negation law "$x^\Diamond \in [X]_Q \iff x \notin [X]_Q$."
(2) X belongs to $\mathcal{C}$ and satisfies $x \in [X]_Q$ or $x^\Diamond \in [X]_Q$ for each element $x \in A$.

Proof due to Takahashi. This follows from Theorem 8.6, because $\big[[X]_Q\big]_Q = [X]_Q$ holds, and $X \in \mathcal{C}$ iff $[X]_Q \in \mathcal{C}$ by Theorem 8.4. □

Theorem 8.7. $\mathcal{B}$-negations and $\mathcal{B}$-complements are related as follows.

(1) Every $\mathcal{B}$-negation $\Diamond$ on A is a $\mathcal{B}$-complement on A.
(2) If a $\mathcal{B}$-negation $\Diamond$ on A exists, then $\mathcal{B} - \{A\} \subseteq \mathcal{D}$. Conversely if $\mathcal{B} - \{A\} \subseteq \mathcal{D}$, then every $\mathcal{B}$-complement on A is a $\mathcal{B}$-negation on A.

Proof. For each element $x \in A$, we have $\{x, x^\Diamond\} \notin \overleftarrow{\mathcal{B} - \{A\}}$, hence $\{x, x^\Diamond\} \notin \mathcal{C}$ by Theorem 8.4. Therefore by Theorem 8.5, $\Diamond$ satisfies the contradiction law. Let $x \in A$ and $X \in \mathcal{P}'A$. Then

$$\big\{B \in \mathcal{B} - \{A\} \;\big|\; \{x\} \cup X \subseteq B\big\} \cup \big\{B \in \mathcal{B} - \{A\} \;\big|\; \{x^\Diamond\} \cup X \subseteq B\big\}$$
$$= \big\{B \in \mathcal{B} - \{A\} \;\big|\; X \subseteq B\big\},$$

and so (8.1) holds by Theorem 8.3. Therefore $\Diamond$ satisfies the excluded middle law. Thus (1) holds. (2) is a consequence of (1) and Theorem 8.6. □

Alternative proof of (1). For each $B \in \mathcal{B} - \{A\}$, define the relation $\preccurlyeq_B$ on A^* as in Theorem 3.27. Then the characteristic mapping 1_B of B is a latticed representation of A on $\mathbb{T}$, and $\preccurlyeq_B$ is equal to the 1_B-validity relation $\preccurlyeq_{1_B}$. Since $\mathbb{T}$ is Boolean, $\preccurlyeq_B$ is a strongly latticed relation by Theorems 3.1 and 3.2. Furthermore, the B-negation law means $1_B(x^\Diamond) = (1_B x)^\Diamond$, where $\Diamond$ on the right-hand side is the complement in $\mathbb{T}$. Therefore by Theorem 3.7, $\preccurlyeq_B$ satisfies the negation laws. Let $\preccurlyeq\; = \bigcap_{B \in \mathcal{B} - \{A\}} \preccurlyeq_B$. Then by Theorem 3.32, $\preccurlyeq$ is also a strongly latticed relation satisfying the negation laws. Therefore by Theorem 3.13, $\preccurlyeq$ satisfies the laws $xx^\Diamond \preccurlyeq y$ and ($\Diamond$6). Since Q is the restriction of $\preccurlyeq$ to $A^* \times A$ by Theorem 6.2, we conclude that $\Diamond$ is a $\mathcal{B}$-complement on A. □

Remark 8.1. If a $\mathcal{B}$-negation on A exists, then it follow from Theorem 8.7 that $(A, \mathcal{B})$ belongs to Class 1 iff $\mathcal{B} = \{A, B\}$ with $B \neq \emptyset$.

Theorem 8.8. Let $X \subseteq A$ and assume that either X is finite or $(A, \mathcal{B})$ belongs to Class 1 or 2. Suppose a $\mathcal{B}$-negation exists. Then in order that $X \in \mathcal{E}$ holds, any one of the following conditions is necessary and sufficient (cf. Corollary 8.6.1).

(1) $[X]_Q$ is the unique $\mathcal{B}$-model of X.
(2) X has a unique $\mathcal{B}$-model.
(3) $[X]_Q \in \mathcal{B} - \{A\}$.

Proof. Assume $X \in \mathcal{E}$. Then Theorem 8.4 shows first that $X \in \mathcal{C}$, next that X has a $\mathcal{B}$-model, and finally that (1) holds. Obviously (1) implies (2). We have $[X]_Q = \bigcap_{X \subseteq B \in \mathcal{B} - \{A\}} B$ by Theorem 8.3. Therefore, if B is the unique $\mathcal{B}$-model of X, then $[X]_Q = B \in \mathcal{B} - \{A\}$ holds. Thus (2) implies (3). If (3) holds, then $X \in \mathcal{E}$ by Theorem 8.7. $\square$

Theorem 8.9. If there exists a $\mathcal{B}$-complement $\Diamond$ on A, then the following conditions are equivalent.

(1) $(A, \mathcal{B})$ belongs to Class 1 or 2.
(2) $\mathcal{D} \subseteq \mathcal{B} - \{A\}$.
(3) $\mathcal{C} \subseteq \overleftarrow{\mathcal{B} - \{A\}}$, that is, every $\mathcal{B}$-consistent subset of A has a $\mathcal{B}$-model.

Proof due to Takaoka. Assume (1). Then $\overline{\mathcal{B}^\cap} = \mathcal{B}^\cap$, an so by Theorem 8.4, $\mathcal{D}$ is equal to the set of the maximal elements of $\mathcal{B}^\cap - \{A\}$. Since maximal elements of $\mathcal{B}^\cap - \{A\}$ belong to $\mathcal{B} - \{A\}$, we conclude that (2) holds. Assume (2). Then $\overleftarrow{\mathcal{D}} \subseteq \overleftarrow{\mathcal{B} - \{A\}}$. Also, since $\{x, x^\Diamond\} \notin \mathcal{C}$ for each $x \in A$ by Theorem 8.5, we have $\mathcal{C} = \overleftarrow{\mathcal{D}}$ by Theorem 8.4. Therefore (3) holds. Finally, we assume (3) and show that (1) holds. By virtue of Theorem 8.3, we only need to show that every element $X \in \mathcal{P}A$ satisfies $[X]_Q = \bigcap_{X \subseteq B \in \mathcal{B}} B$. Obviously $[X]_Q \subseteq \bigcap_{X \subseteq B \in \mathcal{B}} B = \bigcap_{X \subseteq B \in \mathcal{B} - \{A\}} B$, so we will show that every element $y \in \bigcap_{X \subseteq B \in \mathcal{B} - \{A\}} B$ belongs to $[X]_Q$. If $X \subseteq B \in \mathcal{B} - \{A\}$, then $y \in B$, and so $y^\Diamond \notin B$ by Theorem 8.5. Therefore, there does not exist a set B such that $\{y^\Diamond\} \cup X \subseteq B \in \mathcal{B} - \{A\}$, and so (3) yields $[\{y^\Diamond\} \cup X]_Q = A$, hence $y \in [X]_Q$ by (8.1) as desired. $\square$

Remark 8.2. Under the assumption of Theorem 8.9, we have $\overleftarrow{\mathcal{B} - \{A\}} \subseteq \mathcal{C} = \overleftarrow{\overline{\mathcal{B} - \{A\}}}$ by Theorem 8.4. Therefore, the three equivalent conditions in Theorem 8.9 are furthermore equivalent to the following two conditions.

(4) $\mathcal{C} = \overleftarrow{\mathcal{B} - \{A\}}$, that is, a subset of A is $\mathcal{B}$-consistent iff it has a $\mathcal{B}$-model.
(5) $\overleftarrow{\mathcal{B} - \{A\}} = \overleftarrow{\overline{\mathcal{B} - \{A\}}}$, that is, a subset of A has a $\mathcal{B}$-model iff every finite subset of it has a $\mathcal{B}$-model.

If there exists a $\mathcal{B}$-negation on A, then since $\mathcal{B} - \{A\} \subseteq \mathcal{D}$ by Theorem 8.7, the above conditions are furthermore equivalent to the condition $\mathcal{D} = \mathcal{B} - \{A\}$.

Example 8.2. Suppose $(A, \mathcal{F})$ is the functional logical space in PL. Then $\mathcal{F}$ is the set of the binary representations of A. Also, $\mathcal{F}$-negation exists as mentioned in Example 8.1. Therefore by Remark 8.1, $(A, \mathcal{F})$ does not belong to Class 1. Also, if

$X \in \mathcal{D}$, then the characteristic mapping 1_X of X is a binary representation of A, hence $1_X \in \mathcal{F}$. Thus $\mathcal{D} \subseteq \mathcal{B}_{\mathcal{F}} - \{A\}$, and so $(A, \mathcal{F})$ belongs to Class 2.

As for the functional logical space $(A, \mathcal{F})$ in FPL, $\mathcal{C} \subseteq \overleftarrow{\mathcal{B}_{\mathcal{F}} - \{A\}}$ holds as mentioned in Example 7.2, that is, every $\mathcal{F}$-consistent subset of A has an $\mathcal{F}$-model. Therefore, FPL belongs to Class 2.

The class of MPCL depends on the parameter $\mathbb{P}$, which is called the **quantity system** and is defined to be a totally ordered commutative semigroup with identity element, with $\mathbb{Z}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$ being typical ones. Takaoka [13] has shown that MPCL belongs to Class 2 or 3 and that it belongs to Class 2 iff $\mathbb{P}$ is well-ordered and has the largest element.

9. Relations versus operators

Let A be a set. Then an **operator** on A is a transformation φ on $\mathcal{P}A$. If $X \in \mathcal{P}A$ satisfy $\varphi X \subseteq X$, then we call X a **φ-theory** or say that X is closed under φ.

For each increasing operator φ on A, the **finitary core** φ' is the operator on A defined by $\varphi' X = \bigcup_{Y \in \mathcal{P}' X} \varphi Y$ for each $X \in \mathcal{P}A$. Then φ' is the largest among finitary operators ψ which satisfy $\psi \subseteq \varphi$. Also if φ is a closure operator, then so is φ' and its fixture domain is equal to the quasi-finitary closure $\overline{\mathcal{B}}$ of the fixture domain $\mathcal{B}$ of φ. Therefore, $\varphi = \varphi'$ iff $\mathcal{B} = \overline{\mathcal{B}}$.

For each operator φ on A, we define the logic R_φ on A so that $\alpha \, R_\varphi \, y$ iff $\varphi\alpha \ni y$, for each $(\alpha, y) \in A^* \times A$ (cf. Theorem 3.30). Conversely for each logic R on A, we define the operator φ_R on A so that $\varphi_R X$ is equal to the 1st R-descendant X_1 of X, for each $X \in \mathcal{P}A$ (cf. Theorem 4.5). Then φ_R is finitary, and the set of the φ_R-theories is equal to that of the R-theories. The mappings $\varphi \mapsto R_\varphi$ and $R \mapsto \varphi_R$ induce mutually inverting order isomorphisms between the set of the finitary operators on A and that of the logics on A which satisfy the partial weakening law, partial contraction law, and partial exchange law (Sasaki [12]).

Let $(A, \mathcal{B})$ be a logical space. An operator φ on A is said to be **$\mathcal{B}$-sound**, if every element of $\mathcal{B}$ is closed under φ. There exist the largest increasing $\mathcal{B}$-sound operator and the largest finitary $\mathcal{B}$-sound operator on A. Let us denote them by μ and ν. Then $\mu X = \bigcap_{X \subseteq B \in \mathcal{B}} B$ for each $X \in \mathcal{P}A$. Therefore, μ is a closure operator and its fixture domain is equal to the $\cap$-closure $\mathcal{B}^\cap$ of $\mathcal{B}$. Also ν is equal to the finitary core μ' of μ. Therefore, ν is a closure operator and its fixture domain is equal to the quasi-finitary $\cap$-closure $\overline{\mathcal{B}^\cap}$ of $\mathcal{B}$. Thus $\mu = \nu$ iff $\mathcal{B}^\cap = \overline{\mathcal{B}^\cap}$, that is, iff $(A, \mathcal{B})$ belongs to Class 1 or 2 (cf. Theorem 8.3).

Let Q be the largest $\mathcal{B}$-logic on A. Then we can prove $Q = R_\nu$ and $\varphi_Q = \nu$. The former equation gives an alternative proof of Theorem 6.2. The latter shows that the set of the Q-theories is equal to the set $\overline{\mathcal{B}^\cap}$ of the ν-theories. This together with Theorem 6.1 gives an alternative proof of Theorem 6.10, which is in better perspective than the original proof based on Matsuda's mysterious genius.

Operators also give better proofs of certain other theorems in this paper. The notion of completeness and Theorem 7.1 can be restated in terms of operators.

However, the author has not been able to even formulate Theorem 7.12 in terms of operators. This is the reason why operators can not play a principal role in the theory of logical spaces.

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