# Embedding First-Order Predicate Logic in Monophasic Case Logic

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**Abstract** It is shown how the logical system FPL (first-order predicate logic) is embedded in the logical system MPCL (monophasic case logic), which is a prototype of case logic designed for mathematical psychology. Since a logical system is defined to be a pair of syntax and semantics and a logical space is abstracted from semantics, the embedding has its twofold meaning, that is, the embedding of the formal language of FPL in that of MPCL and the embedding of the sentential logical space of FPL in that of MPCL. Consequences of the latter embedding are discussed in the context of the theory of logical spaces. It is also closely related to the embedding of expressible functions.

**Keywords** embedding, expressible functions, first-order predicate logic, formal language, logical space, logical system, mathematical psychology, monophasic case logic, semantics, syntax

The purpose of this paper is to show how the logical system FPL (first-order predicate logic) is embedded in the logical system MPCL (monophasic case logic), which is a prototype of the logical system CL (case logic) introduced by Gomi (2009c) from the viewpoint of MP (mathematical psychology).

Motivations, backgrounds and bibliographic remarks for MP, CL and the embedding have been given in previous papers by Gomi (2009 a, b, c), and this paper is a direct continuation of them. So I will concentrate on the mathematical detail of the embedding using their notation, terminology, definitions and results without reviewing or restating them.

Since a logical system is a pair of syntax and semantics and a logical space is abstracted from semantics as shown by Gomi (09b), the embedding of FPL in MPCL has its twofold meaning. First, each FP language (the formal language of FPL) is embedded in a certain MPC language (the formal language of MPCL). Ultimately, it depends on Theorem 1.1 which clarifies the relationship between USA's (universal sorted algebras) and free algebras. Secondly, the sentential

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functional logical space of FPL is embedded in that of MPCL (Theorems 2.1 and 2.2).

Consequences of the latter embedding will be discussed in the context of the theory of logical spaces by Gomi (09a). In fact, the embedding is derived from that of the denotations (Theorems 2.3 and 2.4), from which the embedding of expressible functions is also derived, that is, a function expressible by a FP language $\mathsf{L}$ is also expressible by the MPC language in which $\mathsf{L}$ is embedded (Theorem 2.5).

# 1 Embedding FP syntax

Here we show how each FP language is embedded in a MPC language.

Let $(\mathsf{A}, \mathsf{T}, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be a MPC language and assume that it satisfies the following two conditions for a positive integer $\mathsf{m}$ or for $\mathsf{m} = \infty$.

(1) The set $\mathsf{K}$ of the cases contains the set $\{\mathsf{n} \in \mathbb{N} \mid \mathsf{n} < \mathsf{m}\}$, and hence $\{1, \ldots, \mathsf{n}\} \subseteq \mathsf{K}$ for all positive integers $\mathsf{n} < \mathsf{m}$.

(2) For some positive integer $\mathsf{n} < \mathsf{m}$, the set $\mathbb{C}_{\{1,\ldots,\mathsf{n}\}}$ of the constant predicates of range $\{1, \ldots, \mathsf{n}\}$ is non-empty.

Note that the operations of $\mathsf{A}$ are indexed by the set

$$\mathsf{L} = \{\breve{o}\mathsf{k}, \mathfrak{r}\mathsf{k}, \wedge, \vee, \Rightarrow, \Diamond, \triangle, \sqcap, \sqcup, \square, \mathfrak{f}, \Omega\mathsf{x} \mid \mathsf{k} \in \mathsf{K},\ \mathfrak{r} \in \mathfrak{Q},\ \mathfrak{f} \in \mathfrak{F},\ \mathsf{x} \in \mathbb{X}_\varepsilon\},$$

although the operations $\sqcap, \sqcup, \square$ do not take part in the embedding.

We begin by introducing two more families of operations on the set $\mathsf{A}$. First let $\mathsf{f} \in \mathbb{C}_{\{1,\ldots,\mathsf{n}\}}$ with $\mathsf{n} < \mathsf{m}$. Then §2.2 of Gomi (09c) shows that we can define the $\mathsf{n}$-ary operation $\mathsf{f}_0$ on $\mathsf{A}$ with $\mathrm{Dom}\,\mathsf{f}_0 = (\mathsf{A}_\varepsilon)^\mathsf{n}$ for the set $\mathsf{A}_\varepsilon$ of the simple nominals by

$$\mathsf{f}_0(\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_\mathsf{n}) = \mathsf{a}_1\, \breve{o}1\, (\mathsf{a}_2\, \breve{o}2\, (\cdots (\mathsf{a}_\mathsf{n}\, \breve{o}\mathsf{n}\, \mathsf{f}) \cdots)) \tag{1.1}$$

for each $(\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_\mathsf{n}) \in (\mathsf{A}_\varepsilon)^\mathsf{n}$, and that the image $\mathrm{Im}\,\mathsf{f}_0$ of $\mathsf{f}_0$ is contained in the set $\mathsf{A}_\emptyset$ of the sentences.

Next as in §3.4 of Gomi (09c), let $\mathsf{x}_0 \in \mathbb{X}_\varepsilon$ be an arbitrary variable simple nominal and define the complex nominal $\mathsf{one} \in \mathsf{A}_\delta$ by

$$\mathsf{one} = (\mathsf{x}_0\, \forall\pi\, \mathsf{x}_0 \triangle)\, \Omega\mathsf{x}_0. \tag{1.2}$$

Then for each element $\mathsf{x} \in \mathbb{X}_\varepsilon$, we can define the unary operations $\forall\mathsf{x}$ and $\exists\mathsf{x}$ on the set $\mathsf{A}$ with $\mathrm{Dom}\,\forall\mathsf{x} = \mathrm{Dom}\,\exists\mathsf{x} = \mathsf{A}_\emptyset$ by

$$\forall\mathsf{x}\,\mathsf{g} = \mathsf{one}\,\forall\pi\,(\mathsf{g}\,\Omega\mathsf{x})\triangle, \qquad \exists\mathsf{x}\,\mathsf{g} = \mathsf{one}\,\exists\pi\,(\mathsf{g}\,\Omega\mathsf{x})\triangle \tag{1.3}$$

for each $\mathsf{g} \in \mathsf{A}_\emptyset$. Both $\mathrm{Im}\,\forall\mathsf{x}$ and $\mathrm{Im}\,\exists\mathsf{x}$ are contained in $\mathsf{A}_\emptyset$.

Using these two families of operations and those in $\mathsf{L}$, define

$$\mathsf{L}_0 = \left\{\mathsf{f}_0, \mathfrak{f}, \wedge, \vee, \Rightarrow, \Diamond, \forall\mathsf{x}, \exists\mathsf{x} \,\middle|\, \mathsf{f} \in \textstyle\bigcup_{\mathsf{n}<\mathsf{m}} \mathbb{C}_{\{1,2,\ldots,\mathsf{n}\}},\ \mathfrak{f} \in \mathfrak{F},\ \mathsf{x} \in \mathbb{X}_\varepsilon\right\}.$$

Regard $\mathsf{A}$ as the algebra equipped with the set $\mathsf{L}_0$ of operations, and let $\mathsf{A}_0$ be the $\mathsf{L}_0$-subalgebra of $\mathsf{A}$ generated by the set $\mathbb{S}_\varepsilon$ of the prime simple nominals:

$$\mathsf{A}_0 = [\mathbb{S}_\varepsilon]_{\mathsf{L}_0}.$$

Then $\mathsf{A}_0 \subseteq \mathsf{A}_\varepsilon \cup \mathsf{A}_\emptyset$, because $\mathbb{S}_\varepsilon \subseteq \mathsf{A}_\varepsilon \cup \mathsf{A}_\emptyset$ and $\mathsf{A}_\varepsilon \cup \mathsf{A}_\emptyset$ is closed under $\mathsf{L}_0$. Also $\mathsf{A}_\varepsilon \subseteq \mathsf{A}_0$, because $\mathsf{A}_\varepsilon = [\mathbb{S}_\varepsilon]_{\mathfrak{F}}$ as was shown in §2.2 of Gomi (09c). Thus

$$\mathsf{A}_0 = \mathsf{A}_\varepsilon \cup (\mathsf{A}_\emptyset \cap \mathsf{A}_0). \tag{1.4}$$

Since the algebra $(\mathsf{A}, \mathsf{L})$ is free over $\mathbb{S}$ by Theorem 2.2 of Gomi (09b), it follows that the algebra $(\mathsf{A}_0, \mathsf{L}_0)$ is also free over $\mathbb{S}_\varepsilon$.

We will furthermore regard $\mathsf{A}_0$ as a USA. As for its sorter, pick a subset

$$\mathsf{T}_0 = \{\varepsilon, \emptyset\}$$

of the sorter $\mathsf{T} = \{\delta, \varepsilon\} \cup \mathcal{P}\mathsf{K}$ of $\mathsf{A}$. Then $\mathsf{T}_0$ is closed under the operations $\mathfrak{f} \in \mathfrak{F}$ and $\wedge, \vee, \Rightarrow, \Diamond$ of $\mathsf{T}$, and so their restrictions to $\mathsf{T}_0$ denoted by the same symbols become operations on $\mathsf{T}_0$ and satisfy the following:

$$\begin{aligned} &\mathrm{Dom}\,\mathfrak{f} = \{\varepsilon\}^{n_\mathfrak{f}}, && \mathfrak{f}(\varepsilon, \ldots, \varepsilon) = \varepsilon, \\ &\mathrm{Dom} \wedge = \mathrm{Dom} \vee = \mathrm{Dom} \Rightarrow = \{\emptyset\}^2, && \emptyset \wedge \emptyset = \emptyset \vee \emptyset = \emptyset \Rightarrow \emptyset = \emptyset, \\ &\mathrm{Dom}\,\Diamond = \{\emptyset\}, && \emptyset^\Diamond = \emptyset. \end{aligned}$$

In addition, for each $x \in \mathbb{X}_\varepsilon$, define the unary operations $\forall x$ and $\exists x$ on $\mathsf{T}_0$ by

$$\mathrm{Dom}\,\forall x = \mathrm{Dom}\,\exists x = \{\emptyset\}, \qquad \forall x\, \emptyset = \exists x\, \emptyset = \emptyset.$$

Also, for each $f \in \mathbb{C}_{\{1,\ldots,n\}}$ with $n < m$, define the $n$-ary operation $f_0$ on $\mathsf{T}_0$ by

$$\mathrm{Dom}\, f_0 = \{\varepsilon\}^n, \qquad f_0(\varepsilon, \ldots, \varepsilon) = \emptyset.$$

Regard $\mathsf{T}_0$ as the algebra equipped with these operations. Then $\mathsf{T}_0$ is an $\mathsf{L}_0$-algebra similar to $\mathsf{A}_0$. As for the sorting, let $\sigma_0$ be the restriction of $\sigma$ to $\mathsf{A}_0$:

$$\sigma_0 = \sigma|_{\mathsf{A}_0}. \tag{1.5}$$

Then $\sigma_0 \mathsf{A}_0 \subseteq \mathsf{T}_0$ by (1.4) and moreover $\sigma_0$ is an $\mathsf{L}_0$-holomorphism of $\mathsf{A}_0$ into $\mathsf{T}_0$. Thus $(\mathsf{A}_0, \mathsf{T}_0, \sigma_0)$ is a sorted algebra, and so $(\mathsf{A}_0, \mathsf{T}_0, \sigma_0, \mathbb{S}_\varepsilon)$ is a USA by Theorem 1.1 below.

Clearly, we may furthermore regard $\mathsf{A}_0$ as the formal language equipped with the syntax $(\mathsf{T}_0, \sigma_0|_{\mathbb{S}_\varepsilon}, \mathbb{S}_\varepsilon, \mathbb{C}_\varepsilon, \mathbb{X}_\varepsilon, \Gamma_0)$ with

$$\Gamma_0 = \left\{ f_0, \mathfrak{f}, \wedge, \vee, \Rightarrow, \Diamond, \forall, \exists \;\middle|\; f \in \textstyle\bigcup_{n<m} \mathbb{C}_{\{1,2,\ldots,n\}},\ \mathfrak{f} \in \mathfrak{F} \right\}.$$

Comparing it with the syntax of the FP languages given in §5.2 of Gomi (09b), we see that $\mathsf{A}_0$ is a FP language with $f_0, \mathfrak{f}$ and $\Diamond$ playing the roles of the predicate symbols, the function symbols and the negation respectively. Since we are assuming $\bigcup_{n<m} \mathbb{C}_{\{1,2,\ldots,n\}} \neq \emptyset$, there exists at least one predicate symbol.

Notice that we may arbitrarily choose $\mathbb{C}_\varepsilon, \mathbb{X}_\varepsilon, \mathfrak{m}, \bigcup_{n<m} \mathbb{C}_{\{1,2,\ldots,n\}}$ and $\mathfrak{F}$ in constructing the MPC language $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ satisfying the conditions (1) and (2). Therefore every FP language is embedded in a MPC language as above.

The following lemma will be used for the proof of Theorem 2.5.

**Lemma 1.1** *Let* $a \in A_0$ *and* $x \in \mathbb{X}$. *Then* $x$ *occurs free in* $a$ *in the free algebra* $(A, L)$ *iff* $x \in \mathbb{X}_\varepsilon$ *and* $x$ *occurs free in* $a$ *in the free algebra* $(A_0, L_0)$.

**Proof** Following the notation of Gomi (09b), if $x$ occurs free in $a$ in the algebra $(A, L)$, we write $x \ll a$. Also, if $x \in \mathbb{X}_\varepsilon$ and $x$ occurs free in $a$ in the algebra $(A_0, L_0)$, we write $x \ll_0 a$.

We argue by induction on the rank $r$ of $a$ in $A_0$. If $r = 0$, or $a \in \mathbb{S}_\varepsilon$, then Theorem 2.2 of Gomi (09b) shows that the following holds:

$$x \ll a \iff x = a \iff x \in \mathbb{X}_\varepsilon \text{ and } x = a \iff x \in \mathbb{X}_\varepsilon \text{ and } \ x \ll_0 a.$$

Therefore assume $r \geq 1$. Then $a$ has a unique word form.

Assume $a = \mathfrak{f}(a_1, \ldots, a_{n_\mathfrak{f}})$ for some $\mathfrak{f} \in \mathfrak{F}$ and $(a_1, \ldots, a_{n_\mathfrak{f}}) \in (A_\varepsilon)^{n_\mathfrak{f}}$. Then Lemma 4.2 of Gomi (09b) shows that $x \ll a$ iff $x \ll a_i$ for some $i \in \{1, \ldots, n\}$, and the induction hypothesis shows that $x \ll a_i$ iff $x \in \mathbb{X}_\varepsilon$ and $x \ll_0 a_i$. Therefore the same lemma shows that $x \ll a$ iff $x \in \mathbb{X}_\varepsilon$ and $x \ll_0 a$. The same argument applies to the word forms $g \wedge h$, $g \vee h$, $g^\diamondsuit$, $g \Rightarrow h$ of $a$ with $g, h \in A_\emptyset \cap A_0$.

Assume $a = f_0(a_1, \ldots, a_n)$ for some $f \in \mathbb{C}_{\{1,2,\ldots,n\}}$ with $n < m$ and $(a_1, \ldots, a_n) \in (A_\varepsilon)^n$. Then (1.1) and Lemma 4.2 of Gomi (09b) shows that $x \ll a$ iff $x \ll a_i$ for some $i \in \{1, \ldots, n\}$, and the induction hypothesis shows that $x \ll a_i$ iff $x \in \mathbb{X}_\varepsilon$ and $x \ll_0 a_i$. Therefore the same lemma shows that $x \ll a$ iff $x \in \mathbb{X}_\varepsilon$ and $x \ll_0 a$.

Assume $a = \forall y\, g$ or $a = \exists y\, g$ for some $y \in \mathbb{X}_\varepsilon$ and $g \in A_\emptyset \cap A_0$. Then (1.2), (1.3) and Lemma 4.2 of Gomi (09b) show that $x \ll a$ iff $y \neq x \ll g$, and the induction hypothesis shows that $y \neq x \ll g$ iff $y \neq x \in \mathbb{X}_\varepsilon$ and $x \ll_0 g$. Therefore the same lemma shows that $x \ll a$ iff $x \in \mathbb{X}_\varepsilon$ and $x \ll_0 a$.

That $(A_0, T_0, \sigma_0, \mathbb{S}_\varepsilon)$ is a USA is a consequence of the following general theorem, which together with Theorem 2.2 of Gomi (09b) clarifies the relationship between USA's and free algebras.

**Theorem 1.1** *Let* $(A, T, \sigma)$ *be a sorted algebra and assume that the algebra* $A$ *has a basis* $S$. *Then* $(A, T, \sigma, S)$ *is a USA.*

**Proof** We only need to show that if $(A', T, \sigma')$ is a sorted algebra and a mapping $\varphi \in S \to A'$ satisfies $\sigma' \varphi = \sigma|_S$, then $\varphi$ is extended to a sort-consistent holomorphism $f \in A \to A'$. The construction of $f$ is similar to that in the final paragraph of the proof of Theorem 2.1 of Gomi (09b) and so will only be outlined.

For each non-negative integer $n$, we will inductively construct the mapping $f_n$ of the $n$-th descendant $S_n$ of $S$ into $A'$ such that $\sigma' f_n = \sigma|_{S_n}$. First, we define $f_0 = \varphi$. Next, let $a \in S_n$ $(n \geq 1)$, and let $(\alpha_\lambda)_{\lambda \in L}$ be the operations of $A$. Then Theorem 2.2 of Gomi (09b) shows that there exists a unique tuple $\lambda, n_j, a_j$ $(j = 1, \dots, k)$ which satisfies $a = \alpha_\lambda(a_1, \dots, a_k)$, $a_j \in S_{n_j}$ $(j = 1, \dots, k)$, and $n = 1 + \sum_{j=1}^{k} n_j$. For each $m \in \{n_1, \dots, n_k\}$, the mapping $f_m \in S_m \to A'$ has been constructed so that $\sigma' f_m = \sigma|_{S_m}$ holds. Let $\alpha'_\lambda$ be the operation of $A'$ indexed by $\lambda$. Then $(f_{n_1} a_1, \dots, f_{n_k} a_k) \in \operatorname{Dom} \alpha'_\lambda$, so we define $f_n a = \alpha'_\lambda(f_{n_1} a_1, \dots, f_{n_k} a_k)$. Thus the mapping $f_n \in S_n \to A'$ has been constructed and satisfies $\sigma' f_n = \sigma|_{S_n}$. Since $A = \coprod_{n=0}^{\infty} S_n$ by Theorem 2.2 of Gomi (09b), we define $f \in A \to A'$ by $f|_{S_n} = f_n$ $(n = 0, 1, \dots)$. Then $f$ is a sort-consistent holomorphism extending $\varphi$.

# 2 Embedding FP semantics

Let $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ and $(A_0, T_0, \sigma_0, \mathbb{S}_\varepsilon, \mathbb{C}_\varepsilon, \mathbb{X}_\varepsilon, \Gamma_0)$ be the MPC language considered in §1 and the FP language constructed within $A$ there. Furthermore, let $(A, \mathcal{W}, (I_W)_{W \in \mathcal{W}})$ and $(A_0, \mathcal{W}_0, (I_{W_0})_{W_0 \in \mathcal{W}_0})$ be the logical systems MPCL and FPL on the languages respectively. Then their definitions in §2 of Gomi (09c) and §5.2 of Gomi (09b) show that both logical systems have a common truth $\emptyset \in T_0$ and the $\emptyset$-parts $W_\emptyset$ and $W_{0,\emptyset}$ of each MPC world $W \in \mathcal{W}$ and each FP world $W_0 \in \mathcal{W}_0$ are the binary lattice $\mathbb{T} = \{0, 1\}$. Furthermore, the $\emptyset$-part $A_{0,\emptyset}$ of $A_0$ is equal to $A_\emptyset \cap A_0$ by (1.4) and (1.5). Thus the logical systems yield the $\mathbb{T}$-valued $\emptyset$-sentential functional logical spaces $(A_\emptyset, \mathcal{F})$ and $(A_\emptyset \cap A_0, \mathcal{F}_0)$ respectively as was shown in §3.8 of Gomi (09b). The main purpose of this section is to prove the following.

**Theorem 2.1** *$(A_\emptyset \cap A_0, \mathcal{F}_0)$ is the restriction of $(A_\emptyset, \mathcal{F})$ in the sense that $\mathcal{F}_0 = \{\varphi|_{A_\emptyset \cap A_0} \mid \varphi \in \mathcal{F}\}$ holds.*

Consequences of this will be discussed in §3 in the context of the theory of logical spaces by Gomi (09a).

Let $\mathcal{V}$ be the set of all triples $(W, \Phi, \nu)$ of a MPC world $W \in \mathcal{W}$, a $\mathbb{C}$-denotation $\Phi$ into $W$ and an $\mathbb{X}$-denotation $\nu$ into $W$. Then each element $(W, \Phi, \nu) \in \mathcal{V}$ yields the composite $\mathrm{pr}_\nu \Phi^* \in A \to W$ of the metadenotation $\Phi^*$ of $A$ into the metaworld $W^{V_{\mathbb{X},W}}$ and the projection $\mathrm{pr}_\nu \in W^{V_{\mathbb{X},W}} \to W$ by $\nu$. Let $\Phi^\nu$ be their composite $\mathrm{pr}_\nu \Phi^*|_{A_\emptyset}$. Then $\Phi^\nu f = (\Phi^* f)\nu \in W_\emptyset = \mathbb{T}$ for all $f \in A_\emptyset$ and $\mathcal{F} = \{\Phi^\nu \mid (W, \Phi, \nu) \in \mathcal{V}\}$. Similarly, let $\mathcal{V}_0$ be the set of all triples $(W_0, \Phi_0, \nu_0)$ of a FP world $W_0 \in \mathcal{W}_0$, a $\mathbb{C}_\varepsilon$-denotation $\Phi_0$ into $W_0$ and an $\mathbb{X}_\varepsilon$-denotation $\nu_0$ into $W_0$ and, for each $(W_0, \Phi_0, \nu_0) \in \mathcal{V}_0$, let $\Phi_0^{\nu_0}$ be the mapping $f \mapsto (\Phi_0^* f)\nu_0$ of $A_\emptyset \cap A_0$ into $W_{0,\emptyset} = \mathbb{T}$. Then $\mathcal{F}_0 = \{\Phi_0^{\nu_0} \mid (W_0, \Phi_0, \nu_0) \in \mathcal{V}_0\}$. Therefore Theorem 2.1 is equivalent to the following.

**Theorem 2.2** *If $(W, \Phi, \nu) \in \mathcal{V}$, then there exists an element $(W_0, \Phi_0, \nu_0) \in \mathcal{V}_0$ such that $\Phi_0^{\nu_0} = \Phi^\nu|_{A_\emptyset \cap A_0}$. Conversely if $(W_0, \Phi_0, \nu_0) \in \mathcal{V}_0$, then there exists an element $(W, \Phi, \nu) \in \mathcal{V}$ such that $\Phi_0^{\nu_0} = \Phi^\nu|_{A_\emptyset \cap A_0}$.*

This will be derived from the two theorems below.

Let $\mathcal{U}$ be the set of all pairs $(W, \Phi)$ of a MPC world $W \in \mathcal{W}$ and a $\mathbb{C}$-denotation $\Phi$ into $W$. Similarly, let $\mathcal{U}_0$ be the set of all pairs $(W_0, \Phi_0)$ of a FP world $W_0 \in \mathcal{W}_0$ and a $\mathbb{C}_\varepsilon$-denotation $\Phi_0$ into $W_0$. Note that each element of $\mathcal{W}$ is a sorted algebra whose sorter is the $\mathsf{M}$-reduct $\mathsf{T}_\mathsf{M}$ of $\mathsf{T}$ for

$$\mathsf{M} = \mathsf{L} \cap \Gamma = \{\breve{o}k, \mathfrak{r}k, \wedge, \vee, \Rightarrow, \Diamond, \triangle, \sqcap, \sqcup, \square, \mathfrak{f} \mid k \in \mathsf{K},\ \mathfrak{r} \in \mathfrak{Q},\ \mathfrak{f} \in \mathfrak{F}\}.$$

Also, each element of $\mathcal{W}_0$ is a sorted algebra whose sorter is the $\mathsf{M}_0$-reduct $\mathsf{T}_{0,\mathsf{M}_0}$ of $\mathsf{T}_0$ for

$$\mathsf{M}_0 = \mathsf{L}_0 \cap \Gamma_0 = \left\{ f_0, \mathfrak{f}, \wedge, \vee, \Rightarrow, \Diamond \;\middle|\; f \in \textstyle\bigcup_{n<m} \mathbb{C}_{\{1,2,\dots,n\}},\ \mathfrak{f} \in \mathfrak{F} \right\}.$$

**Theorem 2.3** *If* $(W, \Phi) \in \mathcal{U}$*, then there exists an element* $(W_0, \Phi_0) \in \mathcal{U}_0$ *which satisfies the following conditions (1)–(5).*

*(1)* $W_0$ *is the union* $W_\varepsilon \cup W_\emptyset$ *of the* $\varepsilon$*-part* $W_\varepsilon$ *and the* $\emptyset$*-part* $W_\emptyset$ *of* $W$*.*

*(2) The sorting of* $W_0$ *is the restriction of that of* $W$*.*

*(3) The operations* $\mathfrak{f} \in \mathfrak{F}$ *and* $\wedge, \vee, \Rightarrow, \Diamond$ *of* $W_0$ *are the restrictions of the operations* $\mathfrak{f} \in \mathfrak{F}$ *and* $\wedge, \vee, \Rightarrow, \Diamond$ *of* $W$ *respectively.*

*(4) For each element* $f \in \mathbb{C}_{\{1,\dots,n\}}$ *with* $n < m$*, the operation* $f_0$ *of* $W_0$ *satisfies* $\mathrm{Dom}\, f_0 = (W_\varepsilon)^n$ *and*

$$f_0(w_1, w_2, \dots, w_n) = w_1\, \breve{o}1\, (w_2\, \breve{o}2\, (\cdots (w_n\, \breve{o}n\, \Phi f) \cdots)) \tag{2.1}$$

*for all* $(w_1, w_2, \dots, w_n) \in (W_\varepsilon)^n$*.*

*(5)* $\Phi_0$ *is the restriction of* $\Phi$*.*

*Conversely if* $(W_0, \Phi_0) \in \mathcal{U}_0$*, then there exists an element* $(W, \Phi) \in \mathcal{U}$ *which satisfies the conditions (1)–(5).*

**Proof** In order to prove the former half, define $W_0 = W_\varepsilon \cup W_\emptyset$. Then the definitions (W8) and (W3) in §2 of Gomi (09c) show that $W_0$ is closed under the operations $\mathfrak{f} \in \mathfrak{F}$ and $\wedge, \vee, \Rightarrow, \Diamond$ of $W$, and so their restrictions to $W_0$ become operations on $W_0$, which we will denote by the same symbols. Next, for each $f \in \mathbb{C}_{\{1,\dots,n\}}$ with $n < m$, since $\Phi f \in W_{\{1,\dots,n\}}$, the definition (W1) in §2 of Gomi (09c) shows that we may define the operation $f_0$ on $W_0$ as in the condition (4). Regard $W_0$ as the algebra equipped with these operations. Thus $W_0$ has become an $\mathsf{M}_0$-algebra similar to $\mathsf{T}_{0,\mathsf{M}_0}$. Let $\rho$ be the sorting of $W$ and define $\rho_0 = \rho|_{W_0}$. Then $\rho_0 W_0 \subseteq \mathsf{T}_0$ and moreover $(W_0, \mathsf{T}_{0,\mathsf{M}_0}, \rho_0)$ is a sorted algebra and the $\varepsilon$-part and the $\emptyset$-part of $W_0$ are equal to $W_\varepsilon$ and $W_\emptyset$ respectively. Furthermore, the definitions in §2 of Gomi (09c) show that $W_\varepsilon \neq \emptyset$ and $W_\emptyset$ is closed under the operations $\wedge, \vee, \Rightarrow$ and $\Diamond$, whose restrictions to $W_\emptyset$ are the meet, join, implication and complement of the binary lattice $W_\emptyset$ $(= \mathbb{T})$. Thus the definitions in §5.2 of Gomi (09b) show that $W_0 \in \mathcal{W}_0$. Since $\Phi\mathbb{C}_\varepsilon \subseteq W_\varepsilon$, the restriction $\Phi_0$ of $\Phi$ to $\mathbb{C}_\varepsilon$ is a $\mathbb{C}_\varepsilon$-denotation into $W_0$.

In order to prove the latter half, let $S$ denote the $\varepsilon$-part of $W_0$. Then there obviously exists a relation on $S$, and there exists a $\mathbb{P}$-measure on $S$ for the quantity system $\mathbb{P}$ of $A$ by Remark 2.1 of Gomi (09c). Therefore there exists an element $W \in \mathcal{W}$ such that $W_\varepsilon = S$ and the action of each $\mathfrak{f} \in \mathfrak{F}$ on $W$ is equal to that on $W_0$, and it necessarily satisfies the conditions (1)–(3). For each $f \in \mathbb{C}_{\{1,\ldots,n\}}$ with $n < m$, the operation $f_0$ of $W_0$ may be regarded as an element of $S^n \to \mathbb{T}$, and $S^n \to \mathbb{T}$ is identified with $(\{1,\ldots,n\} \to S) \to \mathbb{T} = W_{\{1,\ldots,n\}}$. Therefore $\Phi_0$ can be extended to a $\mathbb{C}$-denotation $\Phi$ into $W$ which satisfies $\Phi f = f_0$ for all $f \in \mathbb{C}_{\{1,\ldots,n\}}$ with $n < m$, and Corollary 3.5.2 of Gomi (09c) shows that the $\Phi$ satisfies the condition (4).

**Theorem 2.4** *Assume that elements $(W, \Phi) \in \mathcal{U}$ and $(W_0, \Phi_0) \in \mathcal{U}_0$ satisfy the conditions (1)–(5) of Theorem 2.3. Then, for each $\mathbb{X}$-denotation $\nu$ into $W$, its restriction $\nu|_{\mathbb{X}_\varepsilon}$ to $\mathbb{X}_\varepsilon$ is an $\mathbb{X}_\varepsilon$-denotation into $W_0$ and $(\Phi_0^* a)(\nu|_{\mathbb{X}_\varepsilon}) = (\Phi^* a)\nu$ holds for all elements $a \in A_0$.*

Theorem 2.2 may be derived from Theorems 2.3 and 2.4 by the following argument. First assume $(W, \Phi, \nu) \in \mathcal{V}$. Then $(W, \Phi) \in \mathcal{U}$, and so there exists an element $(W_0, \Phi_0) \in \mathcal{U}_0$ which satisfies the conditions (1)–(5) of Theorem 2.3. Therefore, defining $\nu_0 = \nu|_{\mathbb{X}_\varepsilon}$, we have $(W_0, \Phi_0, \nu_0) \in \mathcal{V}_0$ and $\Phi_0^{\nu_0} = \Phi^\nu|_{A_\emptyset \cap A_0}$. Next assume $(W_0, \Phi_0, \nu_0) \in \mathcal{V}_0$. Then $(W_0, \Phi_0) \in \mathcal{U}_0$, and so there exists an element $(W, \Phi) \in \mathcal{U}$ which satisfies the conditions (1)–(5) of Theorem 2.3. Obviously $\nu_0$ is extended to an element $\nu \in V_{\mathbb{X},W}$, which satisfies $(W, \Phi, \nu) \in \mathcal{V}$ and $\Phi_0^{\nu_0} = \Phi^\nu|_{A_\emptyset \cap A_0}$.

The proof of Theorem 2.4 is as follows. If $\nu \in V_{\mathbb{X},W}$, then since $\nu(\mathbb{X}_\varepsilon) \subseteq W_\varepsilon$, $\nu|_{\mathbb{X}_\varepsilon}$ belongs to $V_{\mathbb{X}_\varepsilon, W_0}$. Therefore we have the mapping $\nu \mapsto \nu|_{\mathbb{X}_\varepsilon}$ of $V_{\mathbb{X},W}$ into $V_{\mathbb{X}_\varepsilon, W_0}$. In order to use this, we need the following general lemma.

**Lemma 2.1** *Let $B$ be a sorted algebra and let $f \in V' \to V$ for non-empty sets $V$ and $V'$. Then there exists a sort-consistent homomorphism $f^B$ of the power algebra $B^V$ into the power algebra $B^{V'}$ which satisfies $f^B c = cf$ for all $c \in B^V$.*

**Proof** Let $U$ be the sorter of $B$. If $c \in B^V$, then there exists an element $u \in U$ such that $c \in V \to B_u$, hence $cf \in V' \to B_u \subseteq B^{V'}$. Therefore, there exists a sort-consistent mapping of $B^V$ into $B^{V'}$ which satisfies $f^B c = cf$ for all $c \in B^V$. In order to prove that $f^B$ is a homomorphism, let $\beta_\lambda, \gamma_\lambda, \gamma'_\lambda$ be the operations of $B, B^V, B^{V'}$ of the same index $\lambda$ and $k$ be their common arity. Then for all $(c_1, \ldots, c_k) \in \operatorname{Dom} \gamma_\lambda$ and all $\nu' \in V'$, we have

$$\begin{aligned}
\big(f^B\big(\gamma_\lambda(c_1, \ldots, c_k)\big)\big)\nu' &= \big(\gamma_\lambda(c_1, \ldots, c_k)\big)(f\nu') \\
&= \beta_\lambda\big(c_1(f\nu'), \ldots, c_k(f\nu')\big) \\
&= \beta_\lambda\big((f^B c_1)\nu', \ldots, (f^B c_k)\nu'\big) \\
&= \big(\gamma'_\lambda(f^B c_1, \ldots, f^B c_k)\big)\nu',
\end{aligned}$$

where the first and the third equalities hold by the definition of $f^B$, and the remaining equalities hold because the projections by $f\nu'$ and $\nu'$ are homomorphisms with respect to $\lambda$. Therefore $f^B\big(\gamma_\lambda(c_1,\dots,c_k)\big) = \gamma'_\lambda(f^B c_1,\dots,f^B c_k)$ for all $(c_1,\dots,c_k) \in \operatorname{Dom}\gamma_\lambda$, and thus the lemma has been proved.

Therefore, there exists a sort-consistent homomorphism $\psi$ of the power algebra $W^{V_{\mathbb{X}_\varepsilon,W_0}}$ into the power algebra $W^{V_{\mathbb{X},W}}$ which satisfies

$$(\psi b)\nu = b(\nu|_{\mathbb{X}_\varepsilon}) \tag{2.2}$$

for all $b \in W^{V_{\mathbb{X}_\varepsilon,W_0}}$ and all $\nu \in V_{\mathbb{X},W}$. The metadenotation $\Phi_0^*$ belongs to $A_0 \to W_0^{V_{\mathbb{X}_\varepsilon,W_0}}$, and the metaworld $W_0^{V_{\mathbb{X}_\varepsilon,W_0}}$ is contained in $W^{V_{\mathbb{X}_\varepsilon,W_0}}$ by the conditions (1) and (2) of Theorem 2.3. Therefore the above equation shows that

$$\big(\psi(\Phi_0^* a)\big)\nu = (\Phi_0^* a)(\nu|_{\mathbb{X}_\varepsilon})$$

for all $a \in A_0$. Thus we have to prove the following for all $a \in A_0$.

$$\psi(\Phi_0^* a) = \Phi^* a. \tag{2.3}$$

We argue by induction on the rank $r$ of $a$ in the free algebra $(A_0, L_0)$. Suppose $r = 0$. Then $a \in \mathbb{S}_\varepsilon = \mathbb{C}_\varepsilon \cup \mathbb{X}_\varepsilon$. If $a \in \mathbb{C}_\varepsilon$, then since $\Phi_0 = \Phi|_{\mathbb{C}_\varepsilon}$, we have $(\psi(\Phi_0^* a))\nu = (\Phi_0^* a)(\nu|_{\mathbb{X}_\varepsilon}) = \Phi_0 a = \Phi a = (\Phi^* a)\nu$ for all $\nu \in V_{\mathbb{X},W}$ by (3.11) of Gomi (09b), hence $\psi(\Phi_0^* a) = \Phi^* a$. If $a \in \mathbb{X}_\varepsilon$, then $(\psi(\Phi_0^* a))\nu = (\Phi_0^* a)(\nu|_{\mathbb{X}_\varepsilon}) = \nu|_{\mathbb{X}_\varepsilon} a = \nu a = (\Phi^* a)\nu$ for all $\nu \in V_{\mathbb{X},W}$ by (3.11) of Gomi (09b), hence $\psi(\Phi_0^* a) = \Phi^* a$. Therefore assume $r \geq 1$. Then $a$ has a unique word form.

If $a = \mathfrak{f}(a_1,\dots,a_{n_\mathfrak{f}})$ for some $\mathfrak{f} \in \mathfrak{F}$ and $(a_1,\dots,a_{n_\mathfrak{f}}) \in (A_\varepsilon)^{n_\mathfrak{f}}$, then since $\Phi_0^*, \psi$ and $\Phi^*$ are homomorphisms with respect to the operation $\mathfrak{f}$, we have

$$\begin{aligned}\psi(\Phi_0^* a) &= \psi\big(\Phi_0^*\big(\mathfrak{f}(a_1,\dots,a_{n_\mathfrak{f}})\big)\big) = \mathfrak{f}\big(\psi(\Phi_0^* a_1),\dots,\psi(\Phi_0^* a_{n_\mathfrak{f}})\big) \\ &= \mathfrak{f}(\Phi^* a_1,\dots,\Phi^* a_{n_\mathfrak{f}}) = \Phi^*\big(\mathfrak{f}(a_1,\dots,a_{n_\mathfrak{f}})\big) = \Phi^* a\end{aligned}$$

by the induction hypothesis. The same argument applies to the word forms $g \wedge h$, $g \vee h$, $g \Rightarrow h$ and $g^\diamondsuit$ of $a$ with $g, h \in A_\emptyset \cap A_0$.

In order to deal with the operation $f_0$ for $f \in \mathbb{C}_{\{1,2,\dots,n\}}$ with $n < m$ and the operations $\forall x$ and $\exists x$ for $x \in \mathbb{X}_\varepsilon$, we need the following facts about how their actions on the metaworld $W_0^{V_{\mathbb{X}_\varepsilon,W_0}}$ are related to $\psi$.

First the operation $f_0$ of $W_0^{V_{\mathbb{X}_\varepsilon,W_0}}$ for $f \in \mathbb{C}_{\{1,2,\dots,n\}}$ with $n < m$ satisfies the following for all $(b_1, b_2, \dots, b_n) \in \big(V_{\mathbb{X}_\varepsilon,W_0} \to W_\varepsilon\big)^n$:

$$\psi\big(f_0(b_1,b_2,\dots,b_n)\big) = \psi b_1 \,\breve{o}1\, (\psi b_2 \,\breve{o}2\, (\cdots(\psi b_n \,\breve{o}n\, \Phi^* f)\cdots)), \tag{2.4}$$

where $f_0(b_1,b_2,\dots,b_n) \in V_{\mathbb{X}_\varepsilon,W_0} \to W_\emptyset$, $\psi b_i \in V_{\mathbb{X},W} \to W_\varepsilon$ for $i \in \{1,\dots,n\}$, $\Phi^* f \in V_{\mathbb{X},W} \to W_{\{1,2,\dots,n\}}$ and both sides belong to $V_{\mathbb{X},W} \to W_\emptyset$. This is verified by the following with $\nu \in V_{\mathbb{X},W}$ arbitrary and $\nu_0 = \nu|_{\mathbb{X}_\varepsilon} \in V_{\mathbb{X}_\varepsilon,W_0}$:

$$\big(\psi\big(f_0(b_1,b_2,\dots,b_n)\big)\big)\nu$$

$$
\begin{aligned}
&= \big(f_0(b_1, b_2, \ldots, b_n)\big)v_0 && \big(\text{by (2.2)}\big)\\
&= f_0(b_1 v_0, b_2 v, \ldots, b_n v_0)\\
&= b_1 v_0\, \breve{o}1\, (b_2 v_0\, \breve{o}2\, (\cdots (b_n v_0\, \breve{o}n\, \Phi f) \cdots)) && \big(\text{by (2.1)}\big)\\
&= (\psi b_1)v\, \breve{o}1\, \big((\psi b_2)v\, \breve{o}2\, \big(\cdots \big((\psi b_n)v\, \breve{o}n\, \Phi f\big) \cdots\big)\big) && \big(\text{by (2.2)}\big)\\
&= (\psi b_1)v\, \breve{o}1\, \big((\psi b_2)v\, \breve{o}2\, \big(\cdots \big((\psi b_n)v\, \breve{o}n\, (\Phi^* f)v\big) \cdots\big)\big)\\
&= \big(\psi b_1\, \breve{o}1\, (\psi b_2\, \breve{o}2\, (\cdots (\psi b_n\, \breve{o}n\, \Phi^* f) \cdots))\big)v,
\end{aligned}
$$

where the second equality holds because the projection $\mathrm{pr}_{v_0} \in W_0^{V_{\mathbb{X}_\varepsilon, W_0}} \to W_0$ is a homomorphism with respect to the operation $f_0$, while the final equality holds because the projection $\mathrm{pr}_v \in W^{V_{\mathbb{X}, W}} \to W$ is a homomorphism with respect to the operations $\breve{o}1, \ldots, \breve{o}n$. The fifth equality holds by (3.11) of Gomi (09b).

Next the operations $\forall x$ and $\exists x$ of ${W_0}^{V_{\mathbb{X}_\varepsilon, W_0}}$ for $x \in \mathbb{X}_\varepsilon$ satisfy the following for all $b \in V_{\mathbb{X}_\varepsilon, W_0} \to W_\emptyset$:

$$
\psi(\forall x\, b) = 1\, \forall\pi\, \big((\psi b)\, \Omega x\big)\triangle, \qquad \psi(\exists x\, b) = 1\, \exists\pi\, \big((\psi b)\, \Omega x\big)\triangle, \tag{2.5}
$$

where 1's in the equations denote the largest element of the power lattice $V_{\mathbb{X}, W} \to W_\delta = V_{\mathbb{X}, W} \to (W_\varepsilon \to \mathbb{T})$. Also, $\forall x\, b, \exists x\, b \in V_{\mathbb{X}_\varepsilon, W_0} \to W_\emptyset$, $\big((\psi b)\, \Omega x\big)\triangle \in V_{\mathbb{X}, W} \to W_{\{\pi\}}$ and both sides belong to $V_{\mathbb{X}, W} \to W_\emptyset$. The equation for $\forall x$ is verified by the following with $v \in V_{\mathbb{X}, W}$ arbitrary:

$$
\begin{aligned}
&\big(\psi(\forall x\, b)\big)v = 1\\
&\iff (\forall x\, b)(v|_{\mathbb{X}_\varepsilon}) = 1 && \big(\text{by (2.2)}\big)\\
&\iff \inf\big\{ b\big((x/w)(v|_{\mathbb{X}_\varepsilon})\big) \bigm| w \in W_\varepsilon \big\} = 1 && \big(\text{by §5.2 of Gomi (09b)}\big)\\
&\iff b\big((x/w)(v|_{\mathbb{X}_\varepsilon})\big) = 1 \text{ for all } w \in W_\varepsilon\\
&\iff (\psi b)\big((x/w)v\big) = 1 \text{ for all } w \in W_\varepsilon && \big(\text{by (2.2)}\big)\\
&\iff \big((\psi b)\, \Omega x\big)v \ni w \text{ for all } w \in W_\varepsilon && \big(\text{by (2.6) of Gomi (09c)}\big)\\
&\iff 1v\, \forall\pi\, \big(\big((\psi b)\, \Omega x\big)v\big)\triangle = 1 && \big(\text{by Theorem 3.9 of Gomi (09c)}\big)\\
&\iff \big(1\, \forall\pi\, \big((\psi b)\, \Omega x\big)\triangle\big)v = 1,
\end{aligned}
$$

where the 1's on the right-hand side of the equations denote the largest element of $\mathbb{T}$, while those on the left-hand side denote the largest element of $V_{\mathbb{X}, W} \to W_\delta$, hence $1v$ is the largest element of $W_\delta$. Also, we have used the fact that the projection $\mathrm{pr}_v$ is a homomorphism with respect to the operations $\triangle$ and $\forall\pi$. The equation for $\exists x$ may be verified similarly.

We can now conclude the proof of (2.3). First if $a = f_0(a_1, \ldots, a_n)$ for $f \in \mathbb{C}_{\{1,2,\ldots,n\}}$ with $n < m$ and for $(a_1, \ldots, a_n) \in (A_\varepsilon)^n$, then

$$
\begin{aligned}
&\psi(\Phi_0^* a)\\
&= \psi\big(\big(\Phi_0^*\big(f_0(a_1, \ldots, a_n)\big)\big)\big)\\
&= \psi\big(f_0(\Phi_0^* a_1, \ldots, \Phi_0^* a_n)\big)\\
&= \psi(\Phi_0^* a_1)\, \breve{o}1\, \big(\psi(\Phi_0^* a_2)\, \breve{o}2\, \big(\cdots \big(\psi(\Phi_0^* a_n)\, \breve{o}n\, \Phi^* f\big) \cdots\big)\big) && \big(\text{by (2.4)}\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \Phi^* a_1 \,\breve{o}1\, (\Phi^* a_2 \,\breve{o}2\, \bigl(\cdots (\Phi^* a_n \,\breve{o}n\, \Phi^* f) \cdots )) && \text{(by induction)}\\
&= \Phi^* \bigl(a_1 \,\breve{o}1\, (a_2 \,\breve{o}2\, (\cdots (a_n \,\breve{o}n\, f) \cdots ))\bigr)\\
&= \Phi^* \bigl(f_0(a_1, a_2, \ldots, a_n)\bigr) && \bigl(\text{by (1.1)}\bigr)\\
&= \Phi^* a,
\end{aligned}
$$

where the second equality holds because $\Phi_0^*$ is a homomorphism with respect to the operation $f_0$, while the fifth equality holds because $\Phi^*$ is a homomorphism with respect to the operations $\breve{o}1, \ldots, \breve{o}n$.

Next if $a = \forall x\, g$ for $x \in \mathbb{X}_\varepsilon$ and $g \in A_\emptyset \cap A_0$, then

$$
\begin{aligned}
&\psi(\Phi_0^* a)\\
&= \psi\bigl(\Phi_0^*(\forall x\, g)\bigr)\\
&= \psi\bigl(\forall x\, (\Phi_0^* g)\bigr)\\
&= 1 \,\forall\pi\, \bigl(\bigl(\psi(\Phi_0^* g)\bigr)\, \Omega x\bigr) \triangle && \bigl(\text{by (2.5)}\bigr)\\
&= 1 \,\forall\pi\, \bigl((\Phi^* g)\, \Omega x\bigr) \triangle && \text{(by induction)}\\
&= (\Phi^* one) \,\forall\pi\, \bigl((\Phi^* g)\, \Omega x\bigr) \triangle && \bigl(\text{by Theorem 3.19 of Gomi (09c)}\bigr)\\
&= \Phi^* \bigl(one \,\forall\pi\, (g\, \Omega x) \triangle\bigr)\\
&= \Phi^* (\forall x\, g) && \bigl(\text{by (1.3)}\bigr)\\
&= \Phi^* a,
\end{aligned}
$$

where the second equality holds because $\Phi_0^*$ is a homomorphism with respect to the operation $\forall x$, while the sixth equality holds because $\Phi^*$ is a homomorphism with respect to the operations $\forall\pi, \triangle$ and $\Omega x$. The proof for $\exists x$ is similar. Thus (2.3) has been proved. This completes the proof of Theorem 2.4.

**Theorem 2.5** *Let $a \in A_0$ and $x_1, \ldots, x_n$ be the free variables of $a$ in $A_0$. Then $x_1, \ldots, x_n$ are free variables of $a$ in $A$, and if elements $(W, \Phi) \in \mathcal{U}$ and $(W_0, \Phi_0) \in \mathcal{U}_0$ satisfy the conditions (1)–(5) of Theorem 2.3, then the functional expression $a^{\Phi_0}(x_1, \ldots, x_n)$ of $a$ under $\Phi_0$ with respect to $x_1, \ldots, x_n$ is equal to the functional expression $a^{\Phi}(x_1, \ldots, x_n)$ of $a$ under $\Phi$ with respect to $x_1, \ldots, x_n$.*

**Proof** Lemma 1.1 shows that $x_1, \ldots, x_n$ are free variables of $a$ in $A$. Both functions are defined on $(W_\varepsilon)^n$, and if $(w_1, \ldots, w_n) \in (W_\varepsilon)^n$, then there exists an element $v \in V_{\mathbb{X}, W}$ such that $v x_i = w_i$ for all $i \in \{1, \ldots, n\}$, hence $a^{\Phi_0}(w_1, \ldots, w_n) = (\Phi_0^* a)(v|_{\mathbb{X}_\varepsilon}) = (\Phi^* a)v = a^{\Phi}(w_1, \ldots, w_n)$ by Theorem 2.4.

# 3 Extending logical spaces

This section is to discuss consequences of Theorem 2.1 in the context of the theory of logical spaces by Gomi (09a).

**Lemma 3.1** *Let $A'$ be a set and $A$ be its subset. Let $R'$ be a logic on $A'$ and $R$ be its restriction to $A^* \times A$. Then the following holds.*

(1) *If* $B'$ *is an* $R'$*-theory in* $A'$*, then* $B' \cap A$ *is an* $R$*-theory in* $A$*.*

(2) *Suppose* $R'$ *is a partially latticed relation. Then for each subset* $X$ *of* $A$*, the* $R$*-closure* $[X]_{(A,R)}$ *of* $X$ *in* $A$ *and the* $R'$*-closure* $[X]_{(A',R')}$ *of* $X$ *in* $A'$ *satisfy* $[X]_{(A,R)} = [X]_{(A',R')} \cap A$*. Therefore the set of the* $R$*-theories in* $A$ *is equal to the set of the intersections* $B' \cap A$ *for the* $R'$*-theories* $B'$ *in* $A'$*.*

**Proof** (1) If elements $x_1, \dots, x_n \in B' \cap A$ and $y \in A$ satisfy $x_1 \cdots x_n \, R \, y$, then since $x_1, \dots, x_n \in B'$ and $x_1 \cdots x_n \, R' \, y$, we have $y \in B'$, hence $y \in B' \cap A$. Thus $B' \cap A$ is an $R$-theory in $A$.

(2) Let $B = [X]_{(A,R)}$ and $B' = [X]_{(A',R')}$. Then since $X \subseteq B' \cap A$, we have $B \subseteq B' \cap A$ by (1). Conversely if $y \in B' \cap A$, then since $R'$ is a partially latticed relation, Theorem 4.5 of Gomi (09a) shows that there exist elements $x_1, \dots, x_n \in X$ such that $x_1 \cdots x_n \, R' \, y$, hence $x_1 \cdots x_n \, R \, y$, and so $y \in B$. Thus $B = B' \cap A$.

A logical space $(A', \mathcal{B}')$ is called an **extension** of a logical space $(A, \mathcal{B})$ if $A \subseteq A'$ and $\mathcal{B} = \mathcal{B}' \cap A = \{B' \cap A \mid B' \in \mathcal{B}'\}$. Altering the terminology of Gomi (09a), we will refer to the $\mathcal{B}$-logics, $\mathcal{B}$-theories and $\mathcal{B}$-core on $A$ as the logics, theories and core of $(A, \mathcal{B})$, and similarly for $(A', \mathcal{B}')$. For instance, the core of $(A, \mathcal{B})$ is the $\mathcal{B}$-core $\bigcap \mathcal{B} = \bigcap_{B \in \mathcal{B}} B$ on $A$.

**Theorem 3.1** *If a logical space* $(A', \mathcal{B}')$ *is an extension of a logical space* $(A, \mathcal{B})$*, then the following holds.*

(1) *The cores* $C$ *and* $C'$ *of* $(A, \mathcal{B})$ *and* $(A', \mathcal{B}')$ *satisfy* $C = C' \cap A$*.*

(2) *The largest logic* $Q$ *of* $(A, \mathcal{B})$ *is the restriction of the largest logic* $Q'$ *of* $(A', \mathcal{B}')$ *to* $A^* \times A$*.*

(3) *The set of the logics of* $(A, \mathcal{B})$ *is equal to the set of the restrictions of the logics of* $(A', \mathcal{B}')$ *to* $A^* \times A$*.*

(4) *Every subset* $X$ *of* $A$ *satisfies* $[X]_{(A,Q)} = [X]_{(A',Q')} \cap A$*. The set of the theories of* $(A, \mathcal{B})$ *is equal to the set of the intersections* $X' \cap A$ *for the theories* $X'$ *of* $(A', \mathcal{B}')$*.*

(5) *The logical space* $(A', \mathcal{B}'^{\cap})$ *is an extension of the logical space* $(A, \mathcal{B}^{\cap})$*, and the logical space* $(A', \overline{\mathcal{B}'^{\cap}})$ *is an extension of the logical space* $(A, \overline{\mathcal{B}^{\cap}})$*.*

(6) *If* $(A', \mathcal{B}')$ *belongs to Class 1, so does* $(A, \mathcal{B})$*. If* $(A', \mathcal{B}')$ *belongs to Class 2, then* $(A, \mathcal{B})$ *belongs to Class 1 or 2.*

**Proof** (1) This is because $C = \bigcap \mathcal{B} = \bigcap (\mathcal{B}' \cap A) = (\bigcap \mathcal{B}') \cap A = C' \cap A$. This follows also from (4) and Theorem 6.4 of Gomi (09a).

(2) This is because the following holds for all $(\alpha, y) \in A^* \times A$ by Theorem 6.2 of Gomi (09a):

$$\alpha \, Q' \, y \iff y \in \textstyle\bigcap_{\alpha \subseteq B' \in \mathcal{B}'} B'$$

$$\iff y \in \bigcap_{\alpha \subseteq B' \in \mathcal{B}'} (B' \cap A)$$
$$\iff y \in \bigcap_{\alpha \subseteq B' \cap A,\ B' \in \mathcal{B}'} (B' \cap A)$$
$$\iff y \in \bigcap_{\alpha \subseteq B \in \mathcal{B}} B$$
$$\iff \alpha \, Q \, y.$$

(3) Let $R$ be a logic of $(A, \mathcal{B})$. Then since $R \subseteq Q$, we have $R \subseteq Q'$ by (2). Therefore $R$ is a logic of $(A', \mathcal{B}')$ by Theorem 6.1 of Gomi (09a), and is obviously the restriction of itself. Conversely let $R'$ be a logic of $(A', \mathcal{B}')$. Then since $R' \subseteq Q'$, its restriction to $A^* \times A$ is contained in $Q$ by (2), and so is a logic of $(A, \mathcal{B})$ by Theorem 6.1 of Gomi (09a).

(4) Since $Q'$ is a partially latticed relation by Theorem 6.3 of Gomi (09a), the former assertion is a consequence of (2) and Lemma 3.1. Theorem 6.1 of Gomi (09a) shows that the theories of $(A, \mathcal{B})$ are the $Q$-theories in $A$ and the theories of $(A', \mathcal{B}')$ are the $Q'$-theories in $A'$. Therefore the latter assertion is also a consequence of (2) and Lemma 3.1.

(5) Since $\mathcal{B} = \mathcal{B}' \cap A$, we immediately have $\mathcal{B}^{\cap} = \mathcal{B}'^{\cap} \cap A$. Theorem 6.10 of Gomi (09a) shows that $\overline{\mathcal{B}^{\cap}}$ is the set of the theories of $(A, \mathcal{B})$ and $\overline{\mathcal{B}'^{\cap}}$ is the set of the theories of $(A', \mathcal{B}')$. Therefore the latter assertion is a consequence of (4).

(6) If $(A', \mathcal{B}')$ belongs to Class 1, then $\overline{\mathcal{B}^{\cap}} = \overline{\mathcal{B}'^{\cap}} \cap A = \mathcal{B}' \cap A = \mathcal{B}$ by (5), and so $(A, \mathcal{B})$ belongs to Class 1. If $(A', \mathcal{B}')$ belongs to Class 2, then $\overline{\mathcal{B}^{\cap}} = \overline{\mathcal{B}'^{\cap}} \cap A = \mathcal{B}'^{\cap} \cap A = \mathcal{B}^{\cap}$ by (5), and so $(A, \mathcal{B})$ belongs to Class 1 or 2.

**Remark 3.1** Takaoka (2009) has shown that the sentential logical space $(A_\emptyset, \mathcal{F})$ of MPCL belongs to Class 2 under certain conditions. The sentential logical space $(A_\emptyset \cap A_0, \mathcal{F}_0)$ of FPL may be embedded in such $(A_\emptyset, \mathcal{F})$ by Theorem 2.1. Thus it follows from Theorem 3.1 that $(A_\emptyset \cap A_0, \mathcal{F}_0)$ also belongs to Class 2 (cf. Theorems 7.6 and 8.9 of Gomi (09a)), although it has been proved independently of them.

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